Counting orbits of discrete subgroups of $O^+(1,n)(\mathbb{R})$ in $\mathbb{R}H^n$

> Let
> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}H^n) \curvearrowright \mathbb{R}H^n$$
> be a discrete subgroup.

We know
$$\Gamma \curvearrowright \mathbb{R}H^n$$

is a proper action so

which means the orbit map
$$\Gamma \to \mathbb{R}H^n$$
$$\gamma \mapsto \gamma(y)$$

gives a bijection
$$\Gamma/\Gamma_x \cong_\Gamma \Gamma\{y\}$$
$$\gamma \mapsto \gamma(y)$$

So counting orbit points is same as counting:

> ➹ Definition. **Orbit counting function**
>
> > Let
> > $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}H^n) \curvearrowright \mathbb{R}H^n$$
> > be a discrete subgroup.
>
> Let $x, y \in \mathbb{R}H^n$ and $r > 0$. Then
> $$N(r,x,y) := \left\|\left\{\gamma \in \Gamma \,\middle|\, \underbrace{d(x,\gamma(y)) < r}_{\iff \gamma(y) \in B_r(x)}\right\}\right\|$$
> is the **orbit counter** that counts *how many points of $\Gamma\{y\}$ is in the ball $B_r(x)$.*

- Let $\gamma_0 \in \Gamma$.

- Then
$$N(r, x, y) = N(r, x, \gamma_0 y)$$

- There is a $r > 0$ small enough such that
$$N(r, x, y) = \begin{cases} 1 \\ |\Gamma_x| \end{cases}$$

the orbit count is bounded above by the exponential in r

Let
$$\Gamma \le O^+(1, n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$

be a discrete subgroup.

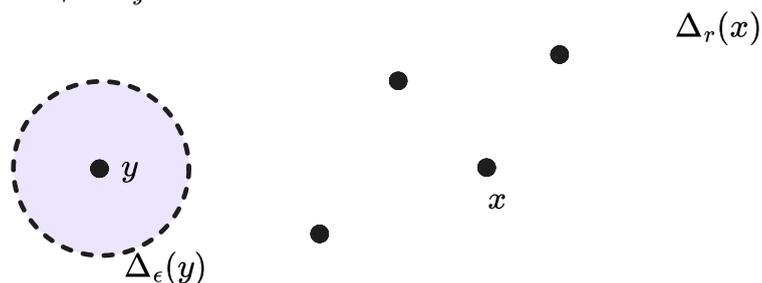**Let $y \in \mathbb{R}\boldsymbol{H}^n$. Then there is a $A_{\Gamma,y} > 0$ such that**
$$\forall x \in \mathbb{R}\boldsymbol{H}^n, N(r, x, y) \le A_{\Gamma,y} e^{r(n-1)}$$

💡 Let $\epsilon > 0$ be small enough so that
$$\{\gamma(\Delta_\epsilon(y)) | \gamma \in \Gamma\}$$

do not overlap unless $\gamma \in \Gamma_y$

by proper

- Then by the volume of balls
$$\mu_{\mathbb{R}\boldsymbol{H}^n}(B_s(o)) = m_{S^{n-1}}(S^{n-1}) \int_{t \in (0,s)} (\sinh t)^{n-1} \mathrm{d}t$$
$$\le \frac{m_{S^{n-1}}(S^{n-1})}{2^{n-1}} \int_{t \in (0,s)} e^{(n-1)t} \mathrm{d}t$$
$$\le \frac{m_{S^{n-1}}(S^{n-1})}{2^{n-1}} \left(\frac{e^{(n-1)s}}{(n-1)} - 1 \right)$$

we have

$$\bigsqcup_{\gamma \in \Gamma : \gamma(y) \in B_r(x)} \gamma(\Delta_\epsilon(y)) \subset \Delta_{r+\epsilon}(x)$$

$$\mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_\epsilon) N(r, x, y) \leq \mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_{r+\epsilon}(x))$$

$$N(r, x, y) \leq \frac{\mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_{r+\epsilon})}{\mu_{\mathbb{R}\boldsymbol{H}^n}(\Delta_\epsilon)}$$

$$= \left(\frac{e^{(n-1)(r+\epsilon)}}{(n-1)} - 1 \right)$$

for lattices, the orbit count is bounded below by exponential in r *eventually*

- 💡 Let Γ be a discrete subgroup of finite covolume, $o \in \mathbb{R}\boldsymbol{H}^n$ be such that $\Gamma_o = \{\mathsf{Id}\}$ and let D be the interior of the Dirichlet domain $D_{\Gamma,o}$, so

$$D := \mathrm{int}(D_{\Gamma,0})$$
$$D(r) := B_r(o) \cap D$$
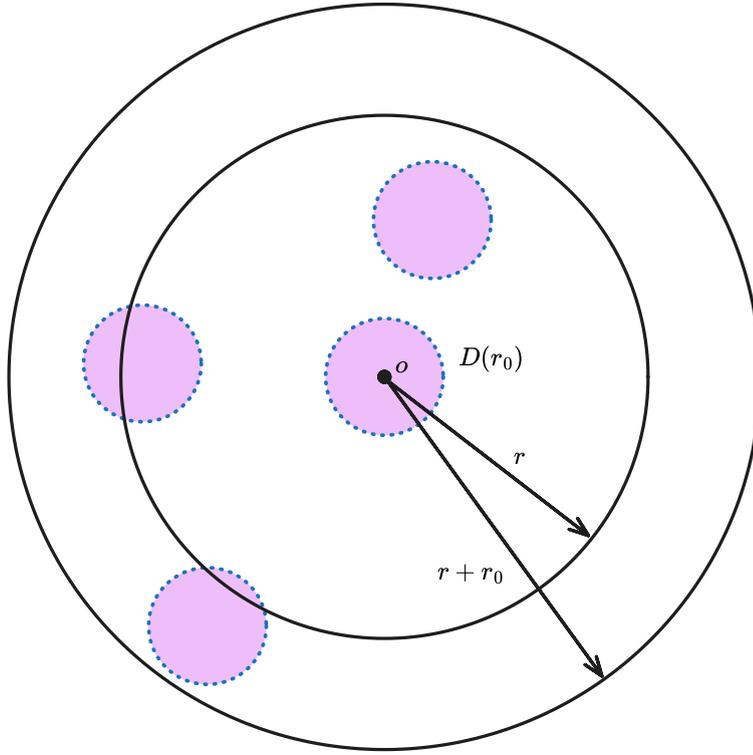
- Then

$$B_r(o) = B_r(o) \cap \Gamma \overline{D}$$
$$= B_r(o) \cap \Gamma(D \sqcup \partial D)$$
$$= B_r(o) \cap \Gamma(D(r_0) \sqcup (D \setminus D(r_0)) \sqcup \partial D)$$

for some $r_0 > 0$.

- Thus

$$\mu(B_r) = \mu(B_r \cap \Gamma D(r_0)) + \mu(B_r \cap \Gamma(D \setminus D(r_0)))$$
$$+ \underbrace{\mu(B_r \cap \Gamma(\partial D))}_{0}$$

-

$$B_r \cap \Gamma D(r_0) = B_r \cap \left(\bigsqcup_{\gamma \in \Gamma} \gamma(D(r_0)) \right)$$

$$\subseteq \bigsqcup_{\gamma \in \Gamma, d(\gamma(y),o) < r+2r_0} \gamma(D(r_0))$$

- For any $y_0 \in D(r_0)$ we have

$$\mu(D(r_0))N(r+r_0, o, y_0) = \mu \left(\bigsqcup_{\gamma \in \Gamma, d(\gamma(y),o) < r+2r_0} \gamma(D(r_0)) \right)$$

$$\geq \mu\left(B_r \cap \Gamma D(r_0)\right)$$

$$= \underbrace{\mu(B_r)}_{=A_1 \int_{t \in (0,r)} (\sinh t)^{n-1} \mathrm{d}t} - \underbrace{\mu(B_r \cap \Gamma(D \setminus D(r_0)))}_{=N(r,o,y)\mu(D \setminus D(r_0))}$$

- Thus for $r > 0$

$$N(r+r_0, o, y_0) \geq A' e^{(n-1)r}$$