Discrete subgroups of $O^+(1,n)$ and hyperbolic manifolds/orbifolds

[1]

> Let
>
> $$\Gamma \leq O^+(1,n)(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{R}\boldsymbol{H}^n) \curvearrowright \mathbb{R}\boldsymbol{H}^n$$
>
> be a discrete subgroup.

category and moduli space of hyperbolic manifolds

Definition. Hyperbolic manifolds

A **$\mathbb{R}$-hyperbolic manifold** is a connected, complete Riemannian manifold with constant sectional curvature $\kappa \equiv 1$.

Let X be a $\mathbb{R}$-hyperbolic n-manifold. Then there is a covering map

$$\begin{array}{ccc} \pi_1(X) \hookrightarrow O^+(1,n)(\mathbb{R}) \curvearrowright & \mathbb{R}\boldsymbol{H}^n \\ & \downarrow \\ & X \end{array}$$

where the Deck transformations act by isometries $\pi_1(X) \hookrightarrow O^+(1,n)(\mathbb{R})$.

Let

$$\mathbb{R}\boldsymbol{H}^n\text{-Riem}$$

be the category of hyperbolic manifolds of dimension n and let

$$\left\{\text{torsion free discrete subgroups of } O^+(1,n)(\mathbb{R})\right\}$$

be the category of torsion free discrete subgroups of $O^+(1,n)(\mathbb{R})$ with morphisms

$$\mathfrak{c}_g : \Gamma_1 \to \Gamma_2$$

being conjugacy homorphisms for $g \in O^+(1,n)(\mathbb{R})$.

Then

$$\begin{aligned} \left\{\text{torsion free discrete subgroups of } O^+(1,n)(\mathbb{R})\right\} &\to \mathbb{R}\boldsymbol{H}^n\text{-Riem} \\ \Gamma &\mapsto {}_{\Gamma}\backslash^{\mathbb{R}\boldsymbol{H}^n} \end{aligned}$$

is a functor, that produces an equivalence of categories?

We have the forgetful functor

$$\mathbb{R}\boldsymbol{H}^n\text{-Riem} \to {}^{n}\mathsf{Man}$$

We have subcategories

$$\mathbb{R}\boldsymbol{H}^n\text{-Riem} \supset \mathbb{R}\boldsymbol{H}^n\text{-FinRiem} \supset \mathbb{R}\boldsymbol{H}^n\text{-cptRiem}$$

a map

$$\mathrm{vol} : \mathbb{R}\boldsymbol{H}^n\text{-}\mathsf{FinRiem} \to [0, \infty)$$

upto isomorphism

We have

$$\{\text{torsion free discrete subgroups of } O^+(1,n)(\mathbb{R})\}_{\cong} \leftrightarrow \mathbb{R}\boldsymbol{H}^n\text{-}\mathsf{Riem}_{\cong} \to {}^{n}\mathsf{Man}_{\cong}$$

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1. Hyperbolic manifolds according to Thurston and Jørgensen - 129-1.pdf ↩