Fuchsian groups $\Gamma <_d PSL(2)(\mathbb{R})$

$\leftrightarrow$

oriented hyperbolic 2-manifold/orbifolds

$\leftrightarrow$

convex polygons in $\mathbb{R}\boldsymbol{H}^2$ equipped with a complete, subproper $\mathrm{Isom}(\mathbb{R}\boldsymbol{H}^2)_\circ$-side-pairing

Definition. **A hyperbolic 2-manifold is a complete, connected Riemannian 2-manifold with curvature $\kappa \equiv -1$. A hyperbolic 2-orbifold is a quotient $\Gamma \backslash \mathbb{R}\boldsymbol{H}^2$ where $\Gamma < \mathrm{Isom}(\mathbb{R}\boldsymbol{H})$ is a discrete subgroup.**

Definition. **A Fuchsian group is a discrete subgroup of $PSL(2, \mathbb{R})$.**

We know $\Gamma < PSL(2, \mathbb{R})$ is a Fuchsian group $\iff$ the action $\Gamma \curvearrowright \mathbb{R}H^2$ is finitely discontinuous.

Proposition: Let Γ be Fuchsian, $H \leq \Gamma$ be of finite index and F be a proper fundamental domain of Γ. Then

$$D = \bigcup_{\gamma \in \Gamma / H} \gamma F$$

is a proper fundamental domain of Γ.
Moreover

$$\mu(D) = [\Gamma : H]\mu(F)$$

if $\mu(F) < \infty$.

Proposition: Let Γ is a Fuchsian group and F is a fundamental domain for Γ.

- If $\gamma \in \Gamma$ is elliptic then $g\gamma g^{-1}$ fixes a point in F for some $g \in \Gamma$.
- If $z \in \mathrm{int}(F)$ then $\Gamma_z = \{I\}$

In particular, $H \leq \Gamma$ is a non-trivial finite subgroup then $gHg^{-1} \leq \Gamma_z$ for some $z \in \partial F$ and $g \in \Gamma$.

Let $\gamma z_0 = z_0$ and $g z_0 \in \Gamma$ Then

$$g\gamma g^{-1}(g z_0) = g z_0$$

Let Γ be cocompact Fuchsian group. Then Γ has no parabolic elements.

Consider

$$\begin{aligned}\varphi : H^2_{\mathrm{U}} &\to [0, \infty) \\ z &\mapsto \min \{d(z, \gamma z) | \gamma \in \Gamma, o(\gamma) = \infty\}\end{aligned}$$

- Then $\varphi(z) = \varphi(\gamma z)$.
- > φ is continuous
- Thus image of φ is a compact subset of $(0, \infty)$.
- So there is a minimum of $\varphi > 0$.
- If $\gamma_0 \in \Gamma$ is parabolic, WLOG $\gamma(z) = z + 1$
- But then $ni \mapsto ni + 1$ has displacement $\leq \frac{1}{n}$.
- This contradicts existence of

Dirichilet region of group acting finitely discontinuously

For all $z \in D_\Gamma(z_0)$ we have

$$|\Gamma \{z\} \cap \partial D_\Gamma(z_0)| < \infty$$

and

$$\sum_{z \in \Gamma \{z\} \cap \partial D_\Gamma(z_0)} \text{internal angle at } z = \frac{2\pi}{|\Gamma_z|}$$

Fuchsian groups	$\Lambda(\Gamma) = \mathbb{R} \cup \{\infty\}$	$\Lambda(\Gamma)$ is a Cantor set	$\lvert\Lambda(\Gamma)\rvert < \infty \iff$ elementary
(geometrically finite) D_Γ has finite sides $\iff$ Γ is finitely generated	$\iff \mu(D_\Gamma) < \infty$		