

Harmonic functions composed with conformal maps

Example

Consider the upper half space $\mathbb{R}^{n-1} \times \mathbb{R}_{>0}$ with (the metric induced from) the standard metric on \mathbb{R}^n . The Laplacian

Definition. We have the Laplace operator

$$\begin{aligned}\Delta : \mathcal{C}^2 &\rightarrow \mathcal{C} \\ \Delta &:= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} (-) \right) + \frac{1}{r^2} \Delta_{S^{n-1}} \\ &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}}\end{aligned}$$

Then

$$u(x) = x_1$$

is *harmonic* and the *inversion* in S^{n-1}

$$\begin{aligned}f : \mathbb{R}^{n-1} \times \mathbb{R}_{>0} &\rightarrow \mathbb{R}^{n-1} \times \mathbb{R}_{>0} \\ x &\mapsto \frac{x}{|x|^2}\end{aligned}$$

is *conformal*. However

$$(u \circ f)(x) = \frac{x_1}{|x|^2}$$

has Laplacian

$$\begin{aligned}\Delta(u \circ f) &= \frac{\Delta(x_1)}{|x|^2} + 2 \left\langle \underbrace{\text{grad}(x_1)}_{e_1}, \text{grad} \left(\frac{1}{|x|^2} \right) \right\rangle + x_1 \Delta \left(\frac{1}{|x|^2} \right) \\ &= 2 \frac{-1}{|x|^4} (2x_1) + x_1 \left(\frac{\partial}{\partial r} \frac{-2}{r^3} + \frac{n-1}{r} \left(-\frac{2}{r^3} \right) \right) \\ &= 2 \frac{-1}{|x|^4} (2x_1) + x_1 \left(\frac{6}{r^4} + \frac{n-1}{r} \left(-\frac{2}{r^3} \right) \right) \\ &= \frac{x_1}{|x|^4} (-4 + 6 - 2n + 2) \\ &= \frac{(4 - 2n)x_1}{|x|^4}\end{aligned}$$

or

$$\begin{aligned}\Delta(u \circ f) &= \left(\frac{\partial}{\partial x_1}\right)^2 \frac{x_1}{|x|^2} && + \sum_{i>1} \left(\frac{\partial}{\partial x_i}\right)^2 \frac{x_1}{|x|^2} \\ &= \frac{\partial}{\partial x_1} \frac{1}{|x|^2} - \frac{\partial}{\partial x_1} \frac{2x_1^2}{|x|^4} && - \sum_{i>1} \frac{\partial}{\partial x_i} \frac{x_1}{|x|^4} (2x_i) \\ &= -\frac{2x_1}{|x|^4} - \frac{4x_1}{|x|^4} + \frac{2x_1^2}{|x|^8} (2|x|^2)(2x_1) && - \frac{2x_1}{|x|^4} (n-1) + \frac{x_1}{|x|^8} \sum_{i>1} (4x_i^2)(2|x|^2) \\ &= \frac{x_1}{|x|^4} (-2 - 4 - 2n + 2) && + 8x_1 \frac{(\sum_{i>1} x_i^2)^2}{|x|^8} \\ &= \frac{(4 - 2n)x_1}{|x|^4}\end{aligned}$$

So it is not harmonic for $n \geq 3$.

[1]