Info

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$\int_{OP} \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-\lambda)}}$ and Weierstrass elliptic function $\wp$

Definition. Weierstrass elliptic function $\wp$

Let $\Lambda \leq \mathbb{C}$ be a rank 2 discrete subgroup. Then the **Weierstrass elliptic $\wp_\Lambda$ function** is

$$\begin{aligned}\wp_\Lambda : \mathbb{C} &\to \mathbb{C}\boldsymbol{P}^1 \\ \wp_\Lambda(z) &:= \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)\end{aligned}$$

and it's (formal) derivative

$$\begin{aligned}\wp'_\Lambda : \mathbb{C} &\to \mathbb{C}\boldsymbol{P}^1 \\ \wp'_\Lambda &= \sum_{\omega \in \Lambda} \frac{-2}{(z-\omega)^3}\end{aligned}$$

Intuition

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \approx \frac{1}{|\omega|^3}$$

For the standard basis $\omega_1, \omega_2 = 1, \tau \in H^2_{\mathrm{U}}$ respectively, we have

$$\wp_\tau := \wp_{\mathbb{Z}\{1,\tau\}}$$

- Let $r > 0$ and consider the closed disk $\overline{D}_r(0)$.
  - The set

    $$\underbrace{\overline{D}_{2r}(0)}_{\text{compact}} \cap \underbrace{\Lambda}_{\text{discrete}}$$

    is finite.
  - $$\begin{aligned}\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| &= \left| \frac{z\left(2 - \frac{z}{\omega}\right)}{\omega^3 \left(1 - \frac{z}{\omega}\right)^2} \right| \\ &\leq \frac{1}{|\omega|^3} 10r \text{ for } z \in \overline{D}_r\end{aligned}$$

The functions $\wp_\Lambda, \wp'_\Lambda$

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converges normally on all compact subsets of $\mathbb{C}$ to odd, even Λ-periodic meromorphic functions with

$$\wp_\Lambda, \wp'_\Lambda : \mathbb{C} \setminus \Lambda \to \mathbb{C}$$

and poles of order 2,3 respectively at each $\omega \in \Lambda$. Moreover, $\wp'_\Lambda$ is the derivative of $\wp_\Lambda$.

Weierstrass elliptic function on the complex tori

By Λ-periodicity, we obtain

$$\wp_\Lambda : {}^{\mathbb{C}}\!/_{\Lambda} \to \mathbb{C}\boldsymbol{P}^1$$

mapping to the elliptic curve

Let

$$\begin{aligned}g_2 &:= 60G_4 \\ g_3 &:= 140G_6 \\ \Delta &:= g_2^3 - 27g_3^2\end{aligned}$$

be defined from the Eisenstein series of order 4 and 6. The Weirstrass elliptic function satisfies

$$(\wp'_\Lambda)(z)^2 = 4\wp_\Lambda(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$$

that is

$$(\wp_\Lambda, \wp'_\Lambda) : \mathbb{C} \setminus \Lambda \to V_{\boldsymbol{A}}(y^2 - (4x^3 - g_2 x - g_3)) \subseteq \mathbb{C}^2$$

which descends and extends to a *biholomorphism*

$$(\wp_\Lambda, \wp'_\Lambda) : {}^{\mathbb{C}}/_{\Lambda} \to V_{\boldsymbol{P}}(y^2 - (4x^3 - g_2 x - g_3)) \subseteq \mathbb{C}\boldsymbol{P}^2$$

Moreover, if $\Lambda = \mathbb{Z}\{\omega_1, \omega_2\}$ the polynomial $4x^3 - g_2 x - g_3$ has distinct roots

$$e_1 := \wp_\Lambda\left(\frac{\omega_1}{2}\right), e_2 := \wp_\Lambda\left(\frac{\omega_2}{2}\right), e_3 := \wp_\Lambda\left(\frac{\omega_1 + \omega_2}{2}\right)$$

$$\Gamma(2) := \ker(PSL(2)(\mathbb{Z}) \to PSL(2)(\mathbb{Z}_2))$$

$$\lambda(A\tau) = \lambda(\tau)$$
$$\lambda(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\tau) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}$$

Current note has 0 direct children and 0 total descendants.

- squishy
  - pow
    - Weierstrass elliptic function $\wp$

And it has 28 siblings.

- squishy
  - pow
    - Angle function
    - Bessel functions
    - $\cos\sqrt{z}$
    - The complex logarithm
    - $\sum_{n\geq 0} z^{n!}$
    - $z^3 - 3z$
    - $\frac{r_0}{1 + \epsilon\cos b\theta}$
    - $\cos\left(\pi\frac{p}{q}\right)$
    - $\frac{\cos x^k}{x}$
    - Elliptic integrals and Jacobi elliptic functions
    - The complex exponential $\exp(z)$
    - $\exp\left(\frac{z+1}{z-1}\right)$
    - e^{-x^2}
    - [Only sum terms that are multiples of N in $\mathbb{C}[[z](https://rupadarshiray.github.io/notes/YchnxQ9SF1RsuKjBntMv8a.]]$
    - $\int \frac{1}{x^2+bx+c}$
    - $\frac{\exp(z)}{1-z}$
    - $\sin x$
    - $\sin(x\sin(x))$
    - $\frac{\sin x}{x}$
    - stock.sin x by xk
    - $\sin x^2$
    - $\sqrt{z}$
    - $\sqrt{z^2 - 1}$
    - Theta functions
    - Weierstrass elliptic function $\wp$
    - Weierstrass' primary factors
    - $x^2\sin\frac{1}{x}$
    - $x + x^2\sin\frac{1}{x}$