Info

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Subgroups of $SL(n,\mathbb{C})$ that are semi-simple

[1]

> **Proposition:** Let $H \leq SL(n,\mathbb{C})$ be a non-Abelian, closed, connected subgroup such that
> $$H \curvearrowright \mathbb{C}^n$$
> is irreducible. Then H is semi-simple.

Let $A \trianglelefteq H$ be a connected, Abelian, normal subgroup of $H \leq SL(n,\mathbb{C})$, and

$$H \to GL(V)$$

is irreducible representation. Then for each continuous $w : A \to \mathbb{C}^\times$ consider

$$V_w := \{v \in V | \forall a \in A, a(v) = w_a v\}$$
$$W := \{w | V_w \neq \{0\}\}$$

- > $W \neq \emptyset$
- > W is finite.
- For $w \in W, h \in H$ let
  $$H \curvearrowright W$$
  $$(hw)(a) := w(h^{-1}ah)$$
- > $$hV_w = V_{h(w)}$$
- But H is connected, implying
  $$H \to \mathrm{AutSet}(W)$$
  has trivial image. Meaning $hV_w = V_w$ for each $w, h \in H$. That is V_w is an H-invariant subspace of $\mathbb{C}^n$.
- Since $H \curvearrowright V$ is irreducible we conclude $V_w = V$ and $W = \{w\}$.
- Thus
  $$a(v) = w(a)v$$
  for all $v \in V$. That is $a = w(a)I$ for all $a \in A$.

- Thus
  $$A \subset \{\lambda I | \lambda \in \mathbb{C}\} \cap SL(n,\mathbb{C}) = \underbrace{\{\lambda I | \lambda \in \mathbb{C}, \lambda^n = 1\}}_{\text{finite}}$$
  - Hence, A is trivial.

> **Proposition:** Let $H \leq SL(n,\mathbb{C})$ be a closed, connected subgroup with finite center $Z(H)$ and
> $$H^\dagger = H$$
> Then H is semi-simple.

Let $W \leq \mathbb{C}^n$ is H-invariant. Then consider $v \in W^\perp$ then

$$\forall w \in W, \langle v, w \rangle = 0$$

- As $HW \leq W, H^\dagger = H$, for any $h \in H$ we have
  $$\forall w \in W, \langle h^\dagger v, w \rangle = 0$$
  $$\forall w \in W, \langle v, hw \rangle = 0$$
  Thus $hw \in W^\perp$. So $W^\perp$ is H-invariant.
- By induction,
  $$\mathbb{C}^n \cong_H \bigoplus_j V_j$$
  such that V_j is H-invariant and $H \curvearrowright V_j$ is irreducible for each j.
- Thus
  $$H \leq \begin{bmatrix} GL(V_1) & & \\ & GL(V_2) & \\ & & \ddots \end{bmatrix}$$
  so
  $$\left\{ \begin{bmatrix} \lambda_1 I & & \\ & \lambda_2 I & \\ & & \ddots \end{bmatrix} \middle| \lambda_1, \ldots, \lambda_n \in \mathbb{C} \right\} \cap H \leq Z(H)$$

- > Let $A \trianglelefteq H$ be a connected, Abelian, normal subgroup of $H \leq SL(n,\mathbb{C})$, and
  > $$H \to GL(V)$$

is irreducible representation. Then for each continuous $w : A \to \mathbb{C}^\times$ consider

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- $W \neq \emptyset$
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- For $w \in W, h \in H$ let
$$H \curvearrowright W$$
$$(hw)(a) := w(h^{-1}ah)$$
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- But H is connected, implying
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has trivial image. Meaning $hV_w = V_w$ for each $w, h \in H$. That is V_w is an H-invariant subspace of $\mathbb{C}^n$.
- Since $H \curvearrowright V$ is irreducible we conclude $V_w = V$ and $W = \{w\}$.
- Thus
$$a(v) = w(a)v$$
for all $v \in V$. That is $a = w(a)I$ for all $a \in A$.

- Thus
$$\underbrace{A}_{\text{connected}} \leq \left\{ \begin{bmatrix} \lambda_1 I & & \\ & \lambda_2 I & \\ & & \ddots \end{bmatrix} \middle| \lambda_1, \dots, \lambda_n \in \mathbb{C} \right\} \cap H \leq \underbrace{Z(H)}_{\text{finite}}$$
Thus, A is trivial.

Current note has 0 direct children and 0 total descendants.

- stem
  - SL C n
    - Subgroups of $SL(n, \mathbb{C})$ that are semi-simple

And it has 1 siblings.

- stem
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    - Subgroups of $SL(n, \mathbb{C})$ that are semi-simple

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1. Dave Witte Morris - Arithmetic groups ↩