> ⓘ Info
>
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> is written (completely with human hands) by Rupadarshi Ray,
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# $SL(2)(\mathbb{Z}) \curvearrowright T^2$

We consider the sub-action of

> **Definition.** Linear maps on the 2-torus
>
> From the module homomorphism
>
> $$(\text{Mat}(2, \mathbb{Z}), \times) \to \left(\text{EndMan}(T^2), \circ\right)$$
>
> we have
>
> $$\begin{aligned} GL(2)(\mathbb{Z}) &\curvearrowright T^2 \\ \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} &\overset{S}{\mapsto} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \end{aligned}$$

by $SL(2)(\mathbb{Z}) \lhd GL(2)(\mathbb{Z})$.

- There are three types of elements of $SL(2)(\mathbb{Z})$

| $\text{tr}(A)^2 - 4$ | $\lvert \text{tr}(A) \rvert$ | **type** | **eigenvalues** $\lambda, \lambda^{-1}$ |
|---|---|---|---|
| $< 0$ | $0, 1$ | elliptic | $\lambda, \lambda^{-1} \in \frac{\text{tr}(A)}{2} \pm i\sqrt{}$ whose norm is 1, so $\lambda \in U(1) \setminus \{-1, 1\}$ |
| $0$ | $2$ | parabolic | $\frac{\text{tr}(A)}{2} \in \{-1, 1\}$ |
| $> 0$ | $\geq 3$ | hyperbolic | $\lambda \in \mathbb{R} \setminus \{-1, 1\}$ |

and their respective diffeomorphism on $T^2$ are thus classified by their *type*.

- **Proposition:**

| **elements, subgroups** | **eigenvalues** $\lambda, \lambda^{-1}$ | $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$ | $SL(2)(\mathbb{Z}) \curvearrowright T^2$ | $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}H$ **by isometries** |
|---|---|---|---|---|
| $\lvert \text{tr} \rvert = 0, 1 \iff$ finite order | $\lambda \in U(1) \setminus \{-1,$ | periodic? | periodic | elliptic isometry (fixes one point) |
| $\lvert \text{tr} \rvert = 2$ | $\lambda \in \{-1, 1\}$ | | reducible; infinite order but not hyperbolic | parabolic isometry (fixes one point in boundary) |
| $\lvert \text{tr} \rvert \geq 3$ | $\lambda \in \mathbb{R} \setminus \{-1, 1\}$ | hyperbolic | (pseudo-)Anosov | hyperbolic isometry (fixes two points in boundary) |

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Current note has 0 direct children and 0 total descendants.

- squishy
  - $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$
  - $SL$
    - $SL(2)(\mathbb{Z})$
- stretchy
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
    - $SL(2)(\mathbb{Z}) \curvearrowright T^2$

And it has 3 siblings.

- stretchy
  - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
    - Arnold cat map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \curvearrowright T^2$
    - $SL(2)(\mathbb{Z}) \curvearrowright T^2$
    - Suspension flows of $SL(2)(\mathbb{Z}) \curvearrowright T^2$