ⓘ Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
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Homogeneous and isotropic Riemannian manifolds

> **Proposition:** Let (M, g) be a Riemannian manifold such that for each $p \in M$ for any $X_p, Y_p \in T_pM$, there exists an isometry $f : (M, g) \to (M, g)$ such that
>
> $$\mathfrak{D}_p f(X_p) = Y_p$$
>
> Then (M, g) is **homogeneous** and isotropic.

Let $p, q \in M$ with a minimizing geodesic arc

$$\gamma : [0, 2a] \to M$$

connecting p, q (not necessarily unique). Then consider the velocity at $\gamma(a)$

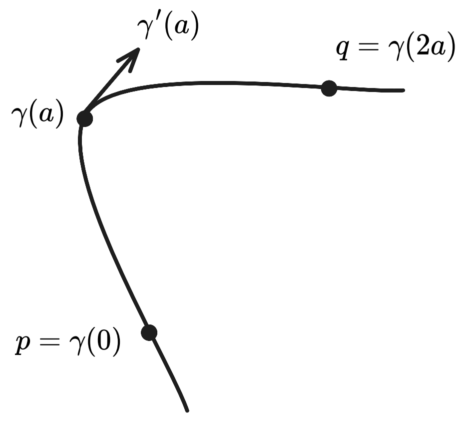
$$\gamma'(a)$$

- By assumption, there is an isometry $f : (M, g) \to (M, g)$ that maps

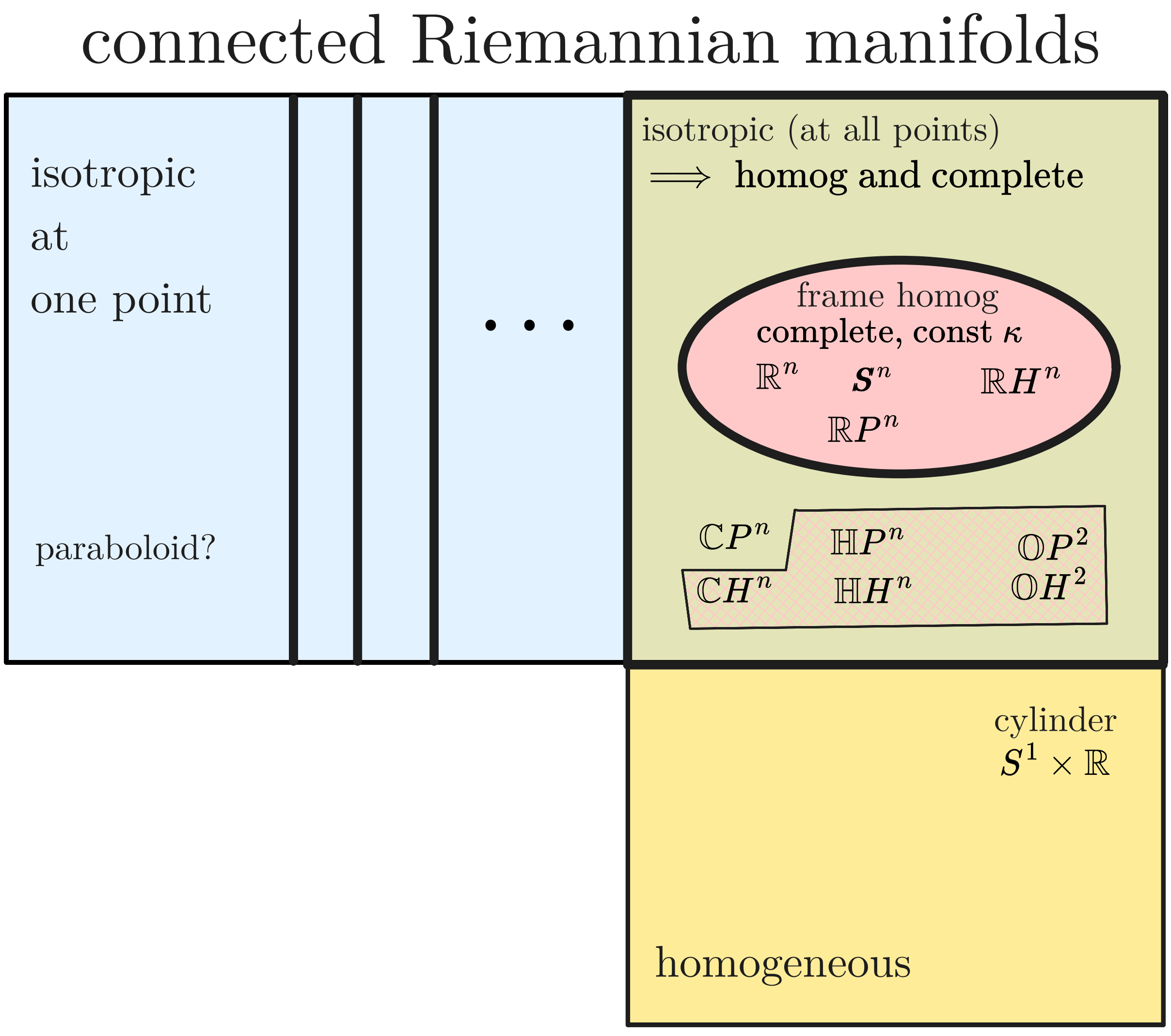
$$\gamma'(a) \mapsto -\gamma'(a)$$

- Then

$$\begin{aligned} f(q) &= f(\exp_{\gamma(a)}(\gamma'(a))) \\ &= \exp_{\gamma(a)}(\mathfrak{D}_a f(\gamma'(a))) \\ &= \exp_{\gamma(a)}(-\gamma'(1)) \\ &= p \end{aligned}$$



- Thus, (M, g) is homogeneous.



> **(Killing-Hopf)** Let (M, g) be a complete, simply connected Riemannian $n \geq 2$-manifold with constant sectional curvature. Then M is isometric to one of
>
> $$\mathbb{R}^n, (S^n, Rg), (\mathbb{R}H^n, Rg)$$

space	$\dim_{\mathbb{R}}$	type	quotient	sectional curvature
$\mathbb{R}^n$	n	flat	$E(n, \mathbb{R})/O(n, \mathbb{R})$	0
S^n	n	compact	$SO(n+1, \mathbb{R})/SO($	$+1$
$\mathbb{R}P^n$	n	compact	$SO(n+1, \mathbb{R})/O($	$+1$
$\mathbb{C}P^n$	$2n$	compact	$SU(n+1, \mathbb{C})/U($	
$\mathbb{H}P^n$		compact	$Sp(n+1)/Sp(n)$	
$\mathbb{O}P^2$		compact		
$\mathbb{R}H^n$	n	non-compact dual of $S^n, \mathbb{R}P^n$	$SO^+(1, n+1, \mathbb{R})/$	-1
$\mathbb{C}H^n$		non-compact dual of $\mathbb{C}P^n$	$SU(1, n+1)/U(n$	
$\mathbb{H}H^n$		non-compact dual of $\mathbb{H}P^n$	$Sp(1, n+1)/Sp(n$	

space	$\dim_{\mathbb{R}}$	type	quotient	sectional curvature
$\mathbb{O}\boldsymbol{H}^2$		non-compact dual of $\mathbb{O}\boldsymbol{P}^2$		

classification

Let M be a connected Riemannian manifold. Then $\mathrm{Isom}(M)$ acts transitively on equidistant pairs of distinct points

$$\iff$$

M Riemannian manifold which is **isotropic** at all points

$$\iff$$

M is isometric to $\mathbb{R}^n$ or a **symmetric space of rank 1**

$$\iff$$

M is isometric to one of

space	$\dim_{\mathbb{R}}$	type	quotient	sectional curvature
$\mathbb{R}^n$	n	flat	$E(n,\mathbb{R})/O(n,\mathbb{R}$	0
S^n	n	compact	$SO(n+1,\mathbb{R})/S$	$+1$
$\mathbb{R}\boldsymbol{P}^n$	n	compact	$SO(n+1,\mathbb{R})/C$	$+1$
$\mathbb{C}\boldsymbol{P}^n$	$2n$	compact	$SU(n+1,\mathbb{C})/U$	
$\mathbb{H}\boldsymbol{P}^n$		compact	$Sp(n+1)/Sp(n$	
$\mathbb{O}\boldsymbol{P}^2$		compact		
$\mathbb{R}\boldsymbol{H}^n$	n	non-compact dual of $S^n, \mathbb{R}\boldsymbol{P}^n$	$SO^+(1,n+1,\mathbb{R}$	-1
$\mathbb{C}\boldsymbol{H}^n$		non-compact dual of $\mathbb{C}\boldsymbol{P}^n$	$SU(1,n+1)/U$	
$\mathbb{H}\boldsymbol{H}^n$		non-compact dual of $\mathbb{H}\boldsymbol{P}^n$	$Sp(1,n+1)/Sp$	
$\mathbb{O}\boldsymbol{H}^2$		non-compact dual of $\mathbb{O}\boldsymbol{P}^2$		

upto a constant scaling.

[2]

[3]

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Complete Riemannian manifolds
        - Homogeneous and isotropic Riemannian manifolds

And it has 14 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Complete Riemannian manifolds
        - Integrating sectional curvature
        - Riemannian manifolds with negative curvature
        - Riemannian manifolds with non-positive sectional curvature
        - Cut locus in a complete Riemannian manifold
        - Exponential map for a complete Riemannian manifold
        - Geodesic completeness $\iff$ metric completeness of Riemannian manifolds
        - Geodesics in homotopy classes
        - Homogeneous Riemannian manifolds
        - Homogeneous and isotropic Riemannian manifolds
        - Poles of a complete Riemannian manifold
        - Riemannian symmetric spaces
        - Uniquely geodesic complete Riemannian manifolds
        - Riemannian homogeneous manifolds where $\exp_p$ is a diffeomorphism
        - Complete Riemannian manifolds without conjugate points

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1. Joseph A. Wolf - Spaces of Constant Curvature (2011, AMS) - 6th ed, p.315 ↩
2. https://math.stackexchange.com/a/4920625 ↩
3. https://math.stackexchange.com/a/4921317 ↩