Info

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$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$

Definition. $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$

The Lie group $SL(2)(\mathbb{R})$ acts smoothly on $\mathbb{R}^2$ by matrix multiplication, with a fixed point $0 \in \mathbb{R}^2$.

- symplectic, that is, area preserving action

$SL(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$

Consider

$$\underbrace{SL(2,\mathbb{R})}_{=:G} \curvearrowright \underbrace{\mathbb{R}^2}_{=:X}$$

There are two orbits

$$\{0\}$$

and

$$\mathbb{R}^2 \setminus \{0\}$$

(disjoint of course).

By orbit-stabilizer theorem

$$\mathbb{R}^2 \setminus \{0\} \cong \frac{G}{\mathcal{U}}$$

where stabilizer is the subgroup

$$\mathcal{U} := \mathrm{stab}_G \begin{pmatrix}1\\0\end{pmatrix} = \left\{\begin{bmatrix}1 & *\\0 & 1\end{bmatrix}\right\} = \exp\left\{\begin{bmatrix}0 & *\\0 & 0\end{bmatrix}\right\} \cong (\mathbb{R},+)$$

$SL(2,\mathbb{R})$-maps

Question

Find all G-invariant maps

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

that is

$$\forall g \in G, x \in \mathbb{R}^2, \phi(gx) = g\phi(x)$$

Because G-orbits will go to G-orbits under such a map we conclude

$$\phi(0) = 0$$

But if we choose where one point of a orbit will go, then we know where the other elements will go

$$\underbrace{g\phi(x)}_{\text{known}} = \underbrace{\phi(gx)}_{\text{unkown}}$$

But to know

$$\phi\begin{pmatrix}1\\0\end{pmatrix}$$

we observe

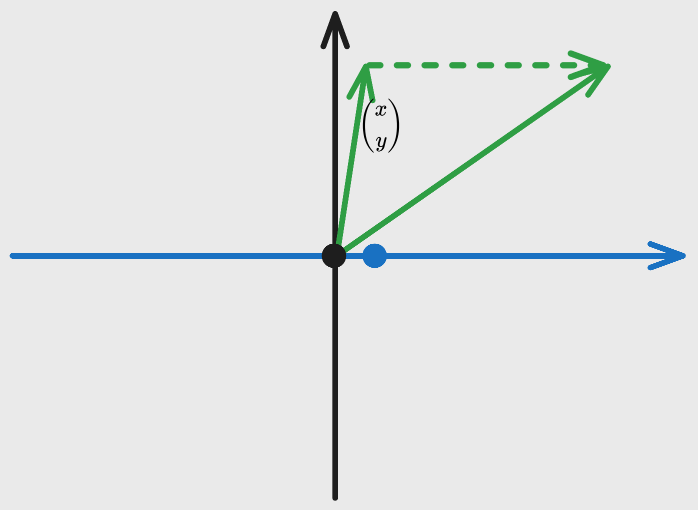
$$g\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \implies \phi\begin{pmatrix}1\\0\end{pmatrix} = \phi(g\begin{pmatrix}1\\0\end{pmatrix}) = g\phi\begin{pmatrix}1\\0\end{pmatrix}$$

means

$$\mathcal{U} \subset \mathrm{stab}_G(\phi\begin{pmatrix}1\\0\end{pmatrix})$$

So we must look at *the orbit structure of $\mathcal{U} \curvearrowright \mathbb{R}^2$* to know what point does it *stabilizes*:

$$\begin{bmatrix}1 & \alpha\\0 & 1\end{bmatrix} \in \mathcal{U} \implies \begin{bmatrix}1 & \alpha\\0 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \alpha\begin{pmatrix}y\\0\end{pmatrix}$$



flow

The dynamics is "tame".

So

$$\mathcal{U} \subset \mathrm{stab}_G(\phi\begin{pmatrix}1\\0\end{pmatrix}) \iff \phi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix}$$

Then

$$\phi(y) = \phi(g\begin{pmatrix}1\\0\end{pmatrix}) = g\phi\begin{pmatrix}1\\0\end{pmatrix} = g\begin{pmatrix}\alpha\\0\end{pmatrix} = \alpha g\begin{pmatrix}1\\0\end{pmatrix} = \alpha y$$

Thus, every G-invariant map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a scaling

$$\phi(x) = \alpha x$$

by some $\alpha \in \mathbb{R}$, which is invertible if $\alpha \in \mathbb{R}^\times$.

Exercise

$$\mathrm{Aut}G\mathsf{Set}\left(\frac{G}{H}\right) \cong \frac{N_G(H)}{H}$$

For our case $H = \mathcal{U}$

$$N_G(\mathcal{U}) = \left\{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right\} \cong \mathbb{R}^\times \times \mathbb{R}$$

and

$$\frac{N_G(\mathcal{U})}{\mathcal{U}} \cong \mathbb{R}^\times$$

$SL(2,\mathbb{Z})$-maps

We may ask

Question

Find all Borel maps

$$\mathbb{R}^2 \to \mathbb{R}^2$$

that commutes with $\Gamma := SL(2,\mathbb{Z})$ defined upto Lebesgue measure-zero sets.

whose answer is

Every Borel measurable $SL(2,\mathbb{Z})$-map $\mathbb{R}^2 \to \mathbb{R}^2$ is essentially a scaling upto Lebesgue measure-zero sets.

But what is special about Γ? The fact that it is a "*lattice*" in G that is:

> **Proposition:** $\Gamma < G$ is discrete and the space
>
> $$G \curvearrowright \frac{G}{\Gamma}$$
>
> has a G-invariant probability measure.

The subgroup

$$A := \left\{\begin{bmatrix}\lambda & \\ & \lambda^{-1}\end{bmatrix} \mid \lambda \in \mathbb{R}^\times\right\} < G = SL(2,\mathbb{R})$$

which acts by $\begin{bmatrix}-1 & \\ & -1\end{bmatrix}$ along with the flow

$$\left\{\exp\left(\alpha\begin{bmatrix}1 & \\ & -1\end{bmatrix}\right) \middle| \alpha \in \mathbb{R}\right\}$$

flow of A

This dynamics is "*tame*" as in the orbits are "locally closed".

"Tame dynamics"	"Wild dynamics"
action by $\mathcal{U}, A$ above on $\mathbb{R}^2$	irrational torus rotations
	$\Gamma \curvearrowright \frac{G}{\mathcal{U}}$
	$\mathcal{U} \curvearrowright \frac{G}{\Gamma}$

There are no $SL(2,\mathbb{R})$-invariant metric on $\mathbb{R}^2$

Every continuous

$$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 &\to [0,\infty) \\ d(x,y) &= d(y,x) \\ d(x,x) &= 0 \end{aligned}$$

satisfying triangle inequality which is $SL(2,\mathbb{R})$-invariant, that is,

$$g \in SL(2,\mathbb{R}) \implies d(gx,gy) = d(x,y)$$

is trivial, as in $d \equiv 0$.

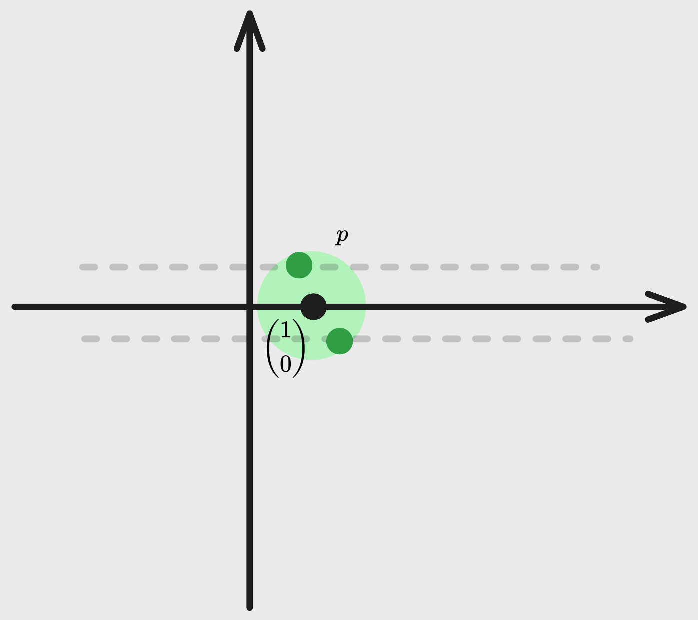
Fix $\epsilon > 0$.

- The restriction $d(\begin{pmatrix}1\\0\end{pmatrix}, -) : \mathbb{R}^2 \to [0,\infty)$ is continuous so preimage of $(0,\epsilon)$ contains a standard ball say B_ϵ.

- Because $\mathcal{U}$ fixes $\begin{pmatrix}1\\0\end{pmatrix}$

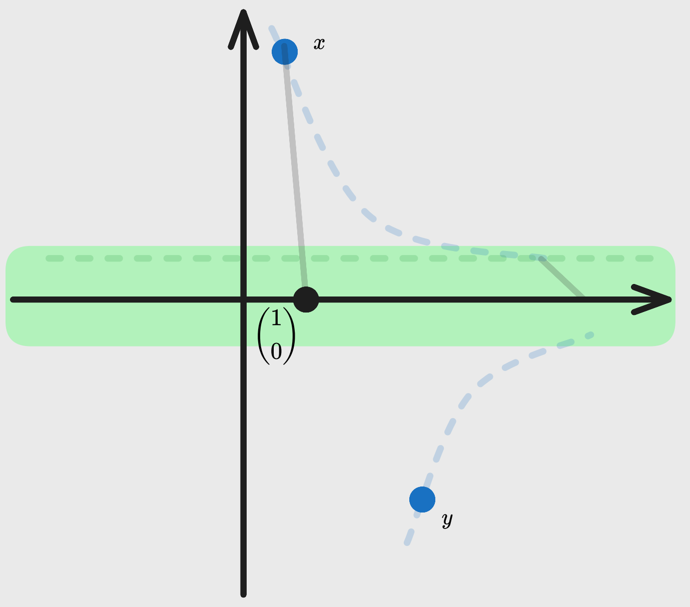
$$d(gp, \begin{pmatrix}1\\0\end{pmatrix}) = d(p, \begin{pmatrix}1\\0\end{pmatrix}) < \epsilon$$

for any $p \in B_\epsilon$.

- 

- Thus the $d(\begin{pmatrix}1\\0\end{pmatrix}, -)$-preimage of $(0, \epsilon)$ actually contains a horizontal strip $U_\epsilon := \mathbb{R} \times (-\epsilon, \epsilon)$.
- Now pick $x, y \in \mathbb{R}^2$ and flow them along A so that they both reach the U_ϵ, so

$$gx \in U_\epsilon$$

- 

- By triangle-inequality

$$d(x, \begin{pmatrix}1\\0\end{pmatrix}) = d(gx, g\begin{pmatrix}1\\0\end{pmatrix}) \le d(gx, \begin{pmatrix}1\\0\end{pmatrix}) + d(g\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}) \le 2\epsilon$$

- By triangle-inequality

$$d(x, y) \le d(x, \begin{pmatrix}1\\0\end{pmatrix}) + d(y, \begin{pmatrix}1\\0\end{pmatrix}) = 4\epsilon$$

- So

$$d(x, y) < 4\epsilon$$

for all $\epsilon > 0$. Thus $d \equiv 0$.

Action of $\mathcal{U}$ on G/Γ is ergodic

Corollary of

Every continuous

$$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty) \\ d(x, y) = d(y, x) \\ d(x, x) = 0 \end{aligned}$$

satisfying triangle inequality which is $SL(2, \mathbb{R})$-invariant, that is,

$$g \in SL(2, \mathbb{R}) \implies d(gx, gy) = d(x, y)$$

is trivial, as in $d \equiv 0$.

: The action of $\mathcal{U}$ on G/Γ is ergodic

Definition. Ergodic actions

An action $G \curvearrowright X$ with an invariant measure class is **ergodic** if either of these is true

- $\forall A \subseteq X$ measurable and invariant A is either a null set or a full measure set
- $L^\infty(X)^G$ is one dimensional
- (if X has a G-invariant probability measure) $L^2(X)^G$ is one dimensional

We observe the action of G on G/Γ is ergodic because the action is transitive?

Let X is a G-space, $\mathcal{U} < G$ and $\Gamma < G$ is a lattice. Then we have a G-invariant measure on G/Γ so we may consider the L^2 space on it. Then

$$L^2(G/\Gamma)^{\mathcal{U}} \cong \mathrm{Hom}_G\left(\frac{G}{\mathcal{U}}, L^2(G/\Gamma)\right)$$

- Every continuous

$$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty) \\ d(x, y) = d(y, x) \\ d(x, x) = 0 \end{aligned}$$

satisfying triangle inequality which is $SL(2, \mathbb{R})$-invariant, that is,

$$g \in SL(2,\mathbb{R}) \implies d(gx, gy) = d(x,y)$$

is trivial, as in $d \equiv 0$.

implying

$$\mathrm{Hom}_G\left(\frac{G}{\mathcal{U}}, L^2(G/\Gamma)\right) \cong L^2(G/\Gamma)^G \cong \mathbb{C}$$

by

> We observe the action of G on G/Γ is ergodic because the action is transitive?

$(\mathbb{R}[X]_2, \{,\}) \to \mathfrak{sl}(2,\mathbb{R})$

polynomial in real variables	polynomial in complex variables	Hamiltonian vector field in complex function form	matrix form of the Hamiltonian vector field	in $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sp}$	flow	type of fixed point at origin
$\frac{\alpha}{2}(x^2+y^2)$	$\frac{\alpha}{2}z\bar{z}$	$i\alpha z$	$\alpha\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$	$\alpha R^{\pi/2}$	rotation	not hyperbolic
αxy	$\frac{\alpha}{4}(z+\bar{z})(z-$	$\frac{i\alpha}{2}(z-\bar{z}-z$	$\alpha\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	αH		hyperbolic
$-\alpha y^2$			$\alpha\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$	αX		not hyperbolic
αx^2			$\alpha\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$	αY		not hyperbolic
general $axy - by^2 +$			$\begin{bmatrix}a & b\\ c & -a\end{bmatrix}$	an arbitrary matrix $aH + bX +$	arbitrary area preserving linear flow	

type of fixed point $0 \in \mathbb{R}^2$ of the flow

$$\begin{bmatrix}a & b\\ c & -a\end{bmatrix} \in \mathfrak{sl}(2,\mathbb{R})$$

det	eigenvalues $\lambda_1, \lambda_2 = \pm\sqrt{-}$	type	Jordan block	exponential	trace of exponential
$\det < 0$	$-x, x \in \mathbb{R} \setminus \{0$	hyperbolic	$\begin{bmatrix}-x & 0\\ 0 & x\end{bmatrix}$	$\begin{bmatrix}e^{-x} & 0\\ 0 & e^{x}\end{bmatrix}$	$e^{-x} + e^{x} > 0$
$\det = 0$	0	parabolic	$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$	2
$\det > 0$	eigenvalues $-ix, ix \in i\mathbb{R}$	elliptic	$\begin{bmatrix}-ix & 0\\ 0 & ix\end{bmatrix}$	$\begin{bmatrix}e^{-ix} & 0\\ 0 & e^{ix}\end{bmatrix}$	$2\cos(x) \geq -2$

✓ elements of $\mathfrak{sl}(2,\mathbb{R})$ with a hyperbolic fixed point

type of fixed point of iterations

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in SL(2)(\mathbb{R})$$

type	elliptic	parabolic	hyperbolic
$\mathrm{tr}^2 - 4$	< 0	$= 0$	> 0
$\lvert\mathrm{tr}\rvert$	< 2	$= 2$	> 2
eigenvalues λ, λ^{-1}	$\lambda \in U(1) \setminus \{-1, 1\}$ $\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm i\frac{\sqrt{-\mathrm{tr}}}{2}$	$\lambda = \lambda^{-1} = \frac{\mathrm{tr}}{2} \in \{-1,$	$\lambda \in \mathbb{R} \setminus \{-1, 1, 0\}$ $\lambda, \lambda^{-1} = \frac{\mathrm{tr}}{2} \pm \frac{\sqrt{\mathrm{tr}^2 -}}{2}$
Jordan block in $GL(2)(\mathbb{C})$	$\begin{bmatrix}e^{-ix} & 0\\ 0 & e^{ix}\end{bmatrix}$	$\pm\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ for $t \in \mathbb{R}$	$\pm\begin{bmatrix}e^{-x} & 0\\ 0 & e^{x}\end{bmatrix}$ for $x \in \mathbb{R}, \pm e^x = \lambda$
elements in $SL(2)(\mathbb{R})$ upto conjugacy	conjugate to rotation matrices $R^\theta \in SO(2)(\mathbb{R})$	conjugate to $\pm\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ for $t \in \mathbb{R}$	as it has two distinct real eigenvalues so it is conjugate to $\begin{bmatrix}\lambda & \\ & \lambda^{-1}\end{bmatrix}$
$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$	elliptic fixed point at 0	parabolic fixed point at 0	hyperbolic fixed point at 0
$SL(2)(\mathbb{R}) \curvearrowright \boldsymbol{P}(\mathbb{R}^2) =$	fixes no line in $\mathbb{R}^2$	fixes exactly one line in $\mathbb{R}^2$ $\iff$ fixes exactly $\begin{bmatrix}1\\ 0\end{bmatrix}$	fixes exactly two lines in $\mathbb{R}^2$
$SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}H^2$	fixes one point $\iff$ conjugate to stabilizer of i which is $SO(2)(\mathbb{R})$	fixes exactly one point in $\partial\mathbb{R}\boldsymbol{H}^n$ $\iff$ conjugate to a transformation only fixing ∞ $\iff$ $z \mapsto z + t$	fixes exactly two points in $\partial\mathbb{R}\boldsymbol{H}^n$ $\iff$ conjugate to $z \mapsto \alpha z$

Current note has 0 direct children and 0 total descendants.

- squishy
    - $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$
    - SL
        - $SL(2)(\mathbb{R})$
- stretchy
    - $GL(2)(\mathbb{Z}) \curvearrowright T^2$
        - $SL(2)(\mathbb{Z}) \curvearrowright T^2$

And it has 30 siblings.

- squishy
    - $SL(2)(\mathbb{R}) \curvearrowright \mathbb{R}^2$
    - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathsf{U}}$
    - $\mathsf{Mat}_{\mathbb{C}}(2)$
    - $A_{r}{}_{\langle z \mapsto \lambda z \rangle} \backslash^{\mathbb{R}\boldsymbol{H}^2_{\mathsf{U}}}$
    - $\mathbb{C} \setminus \{0\}{}_{\langle z \mapsto z+1 \rangle} \backslash^{\mathbb{R}\boldsymbol{H}^2_{\mathsf{U}}}$
    - $\mathbb{C} \setminus \{0, 1\}{}_{PSL(2)(\mathbb{Z})[2]} \backslash^{\mathbb{R}\boldsymbol{H}^2_{\mathsf{U}}}$
    - $\mathbb{R} \ltimes_\theta \mathbb{C}^2$
    - $\boldsymbol{Cl}(\mathbb{R}^n, \langle\ ,\ \rangle_{0,n})$
    - $\mathbb{C}^n$
    - $U(\mathbb{C} \setminus \{0\})$
    - $PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\mathsf{Man}} S^2$
    - $\mathbb{C}P^n$
    - Farey graph, complex and tree
    - GL
    - Heisenberg group
    - Incomplete hyperbolic polygon mod side-pairing
    - $\mathbb{R}\boldsymbol{H}^2, \mathbb{R}H^2_{\mathsf{U}}, D^2$
    - $\mathbb{R}\boldsymbol{H}^n$
    - D_n lattice
    - Matrices
    - $\mathsf{Mat}_{\mathbb{R}}(2)$
    - Mobius endomorphisms
    - Elementary group with finite and infinite sided Dirichlet polyhedra
    - Polynomials
    - $T^*S^1 \ncong_{\mathsf{Sym}} (\mathbb{R}^2 \setminus \{0\}, \mathrm{d}x \wedge \mathrm{d}y)$
    - $Z\left(\sum_{1 \leq j \leq n} z_j^2 - 1\right)(\mathbb{C}) \cong_{\mathsf{Sym}} T^*S^{n-1}$
    - SL
    - SO
    - Seifert–Weber dodecahedral 3-manifold
    - $U(n, \mathbb{C})$