Info

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# Tangent frames on smooth manifolds

Definition. A frame at $p \in M$

A **frame** at a point $p \in M$ is a ordered basis

$$b_p := (b_1, \ldots, b_n)$$

of $T_pM$.

$$\mathscr{F}_pM := \{\text{all frames at } p \in M\}$$

A frame at $p \in M$ is equivalent to a linear bijection between $\mathbb{R}^n$ and $T_pM$ but we won't use this, or give a group structure to a frame.

We shall think of a frame as a choice of ordered basis on $T_pM$, so change of frame at a point, that is change of an ordered basis, is a group action by $GL_{\mathbb{R}}(n)$! This group action is pointwise for all $p \in M$.

Definition. Tangent frame bundle

$$GL_{\mathbb{R}}(n) \curvearrowright \mathscr{F}M \to M$$

The **tangent frame bundle** is a smooth bundle over a smooth manifold with the projection

$$\pi_{\mathscr{F}M} : \mathscr{F}M \to M$$

whose fibres are set of all frames, $\pi_{\mathscr{F}M}^{-1}\{p\} = \mathscr{F}_pM$ which comes with the $GL_{\mathbb{R}}(n)$-action $\mathscr{F}M \times GL_{\mathbb{R}}(n) \to \mathscr{F}M$

$$(b_p, g) \mapsto b_pg := (b_ig)$$

For any smooth $n$-manifold, the tangent frame bundle

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$$(b_p, g) \mapsto b_pg := (b_ig)$$

defined as

$$\mathscr{F}M := \coprod_{p \in M} \mathscr{F}_pM$$

is a smooth manifold and $GL_{\mathbb{R}}(n) \curvearrowright \mathscr{F}X \to X$ is a **smooth principle** $GL_{\mathbb{R}}(n)$**-bundle** over $M$.

Definition. Frames on smooth manifolds

Smooth section of $\mathscr{F}M$ on $U \subseteq M$ is called a frame on $U$.

$\exists$ a global frame on $M \iff M$ is a parallelizable.

Warning

Check

## associated bundles

Define the representation $\rho : GL_{\mathbb{R}}(n) \to GL(\mathbb{R}^n)$

$$g \mapsto \rho(g)(-) := [g](-)$$

then

$$\mathscr{F}M \times_{\rho} \mathbb{R}^n \cong TM$$

Define the representation $\rho^* : GL_{\mathbb{R}}(n) \to GL(\mathbb{R}^n)$

$$g \mapsto \rho^*(g)(-) := [g^T]^{-1}(-)$$

then

$$\mathscr{F}M \times_{\rho^*} \mathbb{R}^n \cong T^*M$$

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Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - Tangent frames on smooth manifolds

And it has 8 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - Canonical bijection of $T_pM$ with equivalence classes of velocities
          - Tangents on smooth manifolds as derivations $\mathcal{C}^\infty(M) \to \mathbb{R}$
          - local basis of a tangent space of a smooth manifold
          - Tangent bundle $TM$ of a smooth manifold $M$
          - Cotangent bundle $T^*M$ on a smooth manifold $M$
          - Comparison of constructions with tangent vectors
          - Tangent frames on smooth manifolds
          - Tangent vector fields on a smooth manifold