Tangent vector fields on a smooth manifold

Rough tangent vector field X on a smooth manifold M is a function

$$X : M \to \coprod_{p \in M} T_pM$$
$$p \mapsto X_p \in T_pM$$

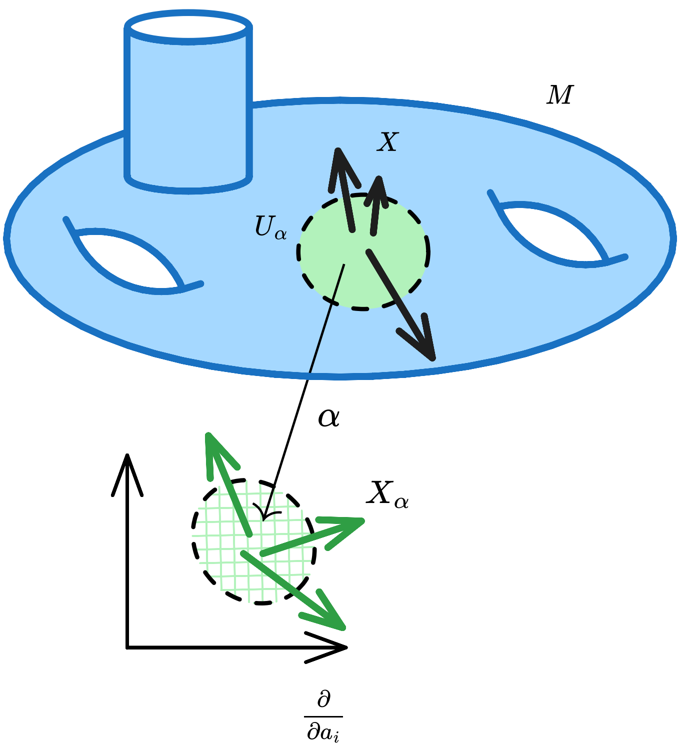
It is *continuous/smooth* if for all charts $(U_\alpha, \underbrace{\alpha}_{U_\alpha \to \mathbb{R}^{\dim M}})$, $p \in U$, X_p written in the coordinate basis for T_pM given by α

$$X_p = \sum_{i=1}^{\dim M} X_{\alpha,i}(p) \frac{\partial}{\partial \alpha_i}\Big|_p$$

makes the map

$$X_\alpha : U_\alpha \to \mathbb{R}^n$$
$$p \mapsto \begin{bmatrix} X_{\alpha,1} \\ X_{\alpha,2} \\ \dots \\ X_{\alpha,\dim M} \end{bmatrix}$$

is a *continuous/smooth* map, which means $X_\alpha \circ \alpha^{-1}$ must be *continuous/smooth* map on $\alpha(U)$.

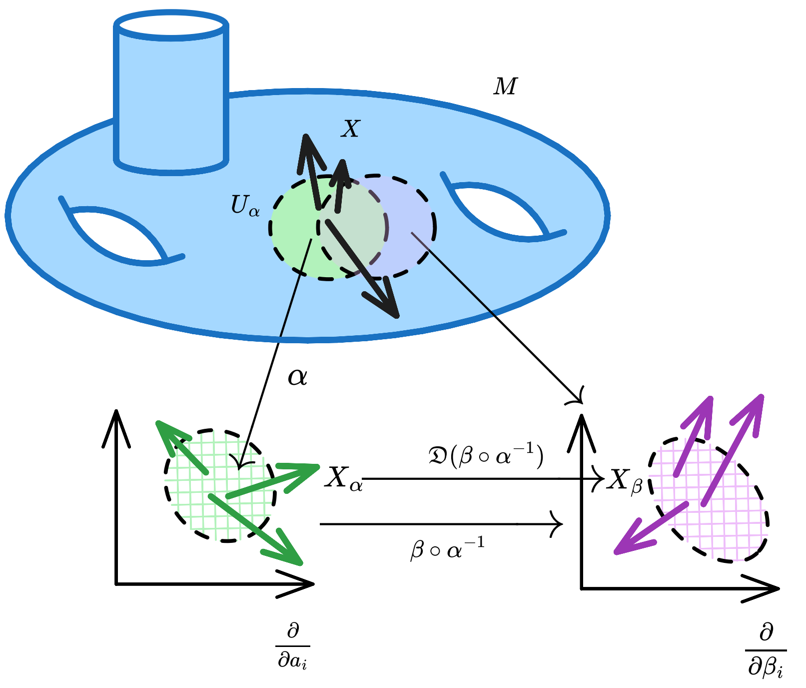


Given the collection of *rough/continuous/smooth* maps

$$(U_\alpha, \alpha) \text{ is a chart } \implies X_\alpha : U_\alpha \to \mathbb{R}^{\dim M}$$

is a vector field on M $\iff$

$$(U_\alpha, \alpha), (U_\beta, \beta) \text{ are two charts on } M \implies \mathfrak{D}(\beta \circ \alpha^{-1})(X_\alpha) = X_\beta \text{ on } U_\alpha \cap U_\beta$$



Definition. Tangent vector field on a smooth manifold

A **smooth/continuous/rough tangent vector field** on a smooth manifold M (with/out boundary) is a *section* of the projection $TM \to M$, that is, a *smooth/continuous/arbitrary* map

$$X : M \to TM$$

satisfying

$$\pi_{TM} \circ X = \mathsf{Id}_M$$

space of all tangent vector fields

Definition. Smooth sections of TM

$$\Gamma(TM) := \{\text{ smooth sections } M \to TM\}$$

- *addition* $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

$$(X + Y)(p) := X(p) + Y(p)$$

- $\mathbb{R}$*-multiplication* $\mathbb{R} \times \Gamma(TM) \to \Gamma(TM)$

$$(\alpha X)(p) := \alpha X(p)$$

- $\mathcal{C}^\infty(M)$*-multiplication* $\mathcal{C}^\infty(M) \times \Gamma(TM) \to \Gamma(TM)$

$$(fX)(p) := f(p)X(p)$$

$(\Gamma(TM), +, \ )$ is a $\mathcal{C}^\infty(M)$-module and a $\mathbb{R}$-vector space

- Vector field on smooth manifold M as derivation of $\mathcal{C}^\infty(M)$
- more operations on $\Gamma(TM)$
  - **Definition.** $[X, Y]$

    Given $X, Y \in \Gamma(TM)$, obtain the **Lie bracket of X and Y**

    $$[X, Y] : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

    defined by

    $$[X, Y](f) := XYf - YXf$$
  - **Definition.** Action of vector fields on $\mathcal{C}^\infty(M)$

    If $X \in \Gamma(TM), f \in \mathcal{C}^\infty(M)$ define the multiplication

    $$\Gamma(TM) \times \mathcal{C}^\infty(M) \to \Gamma(TM)$$

    such that $(X, f) \mapsto Xf$ where

    $$(Xf)(p) := X_p f$$
  - Exponential of vector fields
- structure on $\Gamma(TM)$
  - $\mathbb{R}$-Lie algebra

    $$(\Gamma(TM), +, \ , [\ ,\ ])$$
  - $\mathcal{C}^\infty(M)$-module
- Index of a vector field

Current note has 4 direct children and 4 total descendants.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - Tangent vector fields on a smooth manifold
            - Vector field on smooth manifold M as derivation of $\mathcal{C}^\infty(M)$
            - Exponential of vector fields
            - Index of a vector field
            - Lie bracket of two smooth vector fields on manifold

And it has 8 siblings.

- structures based on sets
  - manifolds
    - smooth $\mathbb{R}$-manifold
      - Tangents on a smooth manifold
        - Tangent space of a smooth manifold
          - Canonical bijection of T_pM with equivalence classes of velocities
          - Tangents on smooth manifolds as derivations $\mathcal{C}^\infty(M) \to \mathbb{R}$
          - local basis of a tangent space of a smooth manifold
          - Tangent bundle TM of a smooth manifold M
          - Cotangent bundle T^*M on a smooth manifold M
          - Comparison of constructions with tangent vectors
          - Tangent frames on smooth manifolds
          - Tangent vector fields on a smooth manifold