Info

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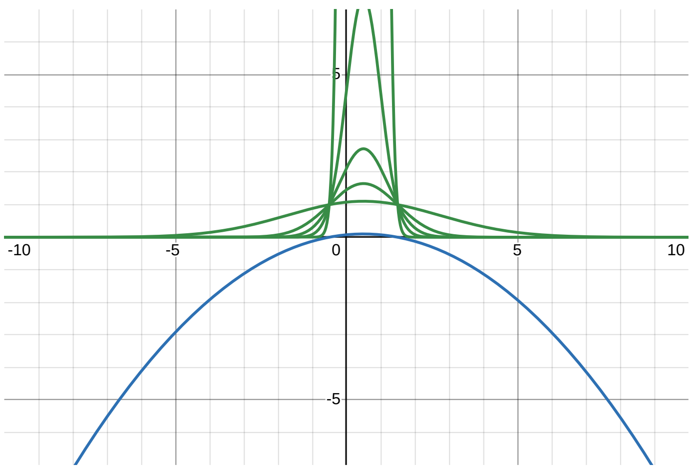
Laplace's method for approximating integral of e^{Mf} for large M

Let $f : [a,b] \to \mathbb{R}$ have a unique global maximum at $x_0 \in (a,b)$.

Example

$$f(x) = \frac{1}{10}\left(1 - \left(x - \frac{1}{2}\right)^2\right)$$



Naive calculations

$$f(x) = f(x_0) + \frac{(x-x_0)^2}{2} f''(x_0) + O(x-x_0)^3$$

Then

$$\int_{[a,b]} e^{Mf} = e^{Mf(x_0)} \int_{[a,b]} e^{M\frac{(x-x_0)^2}{2} f''(x_0)} \int_{[a,b]} e^{MO(x-x_0)^3}$$

where

$$\int_{[a,b]} e^{M\frac{(x-x_0)^2}{2} f''(x_0)} = \sqrt{\frac{2\pi}{-Mf''(x_0)}}$$

by Gaussian integral.

Let $f \in \mathcal{C}^2[a,b]$ and there exists a unique global maximum $x_0 \in (a,b)$ such that

$$f(x_0) = \max_{x \in [a,b]} f(x),\ f''(x_0) < 0$$

then

$$\lim_{M \to \infty} \frac{\int_{[a,b]} e^{Mf}}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{-Mf''(x_0)}}} = 1$$

- **Lower bound**
  - $\forall \epsilon > 0, \exists \delta > 0$ such that

    $$\begin{aligned} x \in B_\delta(x_0) &\implies f''(x_0) - f''(x) \le \epsilon \\ &\implies f''(x) \ge f''(x_0) - \epsilon \end{aligned}$$

  - Because the above inequality is true on $B_\delta(x_0)$ using

    (Mean value form of Taylor's theorem) Let $f : [a,b] \to \mathbb{R}$ and a positive integer $n \in \mathbb{Z}_{>0}$ with $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). Then there exists $c \in (\alpha, x) \subseteq [a,b]$ such that

    $$f(x) = \sum_{k=0}^{n-1}\left(\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k\right) + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n$$

    (where the c depends on x).

    and same δ gives

    $$x \in B_\delta(x_0) \implies f(x) \ge f(x_0) + \frac{(x-x_0)^2}{2}(f''(x_0) - \epsilon)$$

  - Because $e^x > 0$ we have

    $$\int_{[a,b]} e^{Mf} \ge \int_{B_\delta(x_0)} e^{Mf}$$

    which along with

    $$\begin{aligned} &\int_{B_\delta(x_0)} e^{Mf(x_0) + \frac{M(x-x_0)^2}{2}(f''(x_0)-\epsilon)} \\ &= e^{Mf(x_0)}\sqrt{\frac{2\pi}{-M(f''(x_0)-\epsilon)}} \end{aligned}$$

    gives

    $$\epsilon > 0 \implies \frac{\int_{[a,b]} e^{Mf}}{e^{\alpha f^n Mf(x_0)}\sqrt{\frac{2\pi}{-Mf''(x_0)}}} \ge \sqrt{\frac{f''(x_0)}{f''(x_0) - \epsilon}}$$

- **Upper bound**

- Let $f''(x_0)+\epsilon<0$ then by

> **(Mean value form of Taylor's theorem)** Let $f:[a,b]\to\mathbb{R}$ and a positive integer $n\in\mathbb{Z}_{>0}$ with $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). Then there exists $c\in(\alpha,x)\subseteq[a,b]$ such that
>
> $$f(x)=\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k\right)+\frac{f^{(n)}(c)}{n!}(x-\alpha)^n$$
>
> (where the c depends on x).

Taylor's theorem, we find $\delta>0$ such that

$$x\in B_\delta(x_0)\implies f(x)\le f(x_0)+\frac{(x-x_0)^2}{2}(f''(x_0)+\epsilon)$$

- Because $a,b\in\mathbb{R}$ we have $\eta>0$ such that

$$x\in[a,b]\setminus B_\delta\implies f(x)\le f(x_0)-\eta$$

- Now

$$\begin{aligned}\int_{[a,b]}e^{Mf}&\le\left(\int_{[a,b]\setminus B_\delta}+\int_{B_\delta(x_0)}\right)e^{Mf}\\&\le(b-a)e^{M(f(x_0)-\eta)}+\underbrace{\int_{B_{\delta(x_0)}}e^{Mf(x_0)+\frac{M(x-x_0)^2}{2}(f''(x_0)+\epsilon)}}_{\sqrt{\frac{2\pi}{-M(f''(x_0)+\epsilon)}}}\end{aligned}$$

so we have

$$\begin{aligned}\epsilon>0\implies&\frac{\int_{[a,b]}e^{Mf}}{e^{Mf(x_0)}\sqrt{\frac{2\pi}{-Mf''(x_0)}}}\\&\le(b-a)e^{-M\eta}\sqrt{-M\frac{f''(x_0)}{2\pi}}+\sqrt{\frac{-f''(x_0)}{-f''(x_0)-\epsilon}}\end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n,S^n,\mathbb{C}^n$ and its quotients $\Gamma\backslash^{\mathbb{R}^n},\Gamma\backslash^{\mathbb{C}^n}$
    - int
      - Laplace's method for approximating integral of e^{Mf} for large M

And it has 4 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n,S^n,\mathbb{C}^n$ and its quotients $\Gamma\backslash^{\mathbb{R}^n},\Gamma\backslash^{\mathbb{C}^n}$
    - int
      - Integration by parts
      - Laplace's method for approximating integral of e^{Mf} for large M
      - Integral on bounded open subsets of $\mathbb{R}^n$ with $\mathcal{C}^1$ boundary
      - Riemann-Darboux and Riemann-Stieltjes integrals