> Info
>
> This note, that was found here, as a part of a collection
> is written (completely with human hands) by Rupadarshi Ray,
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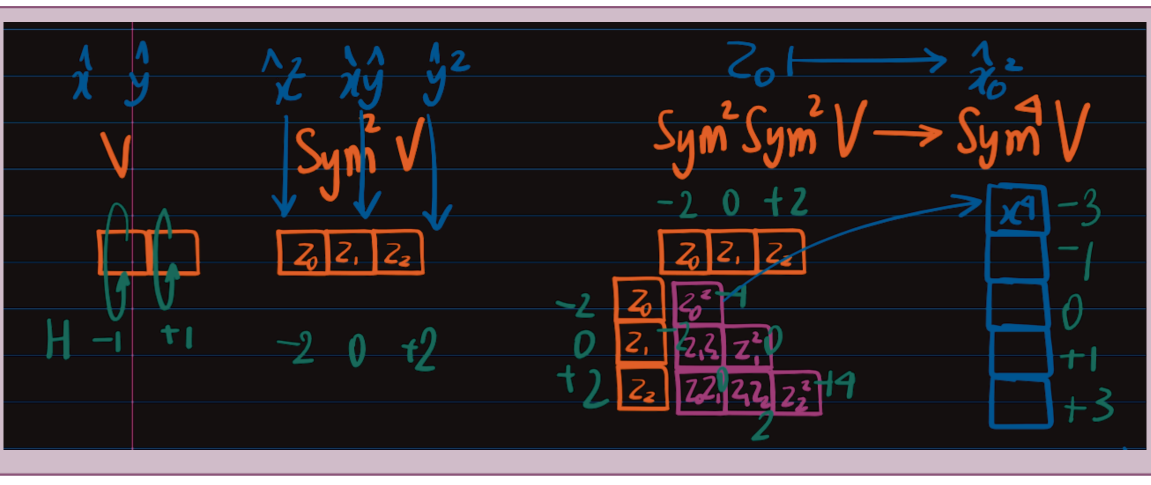
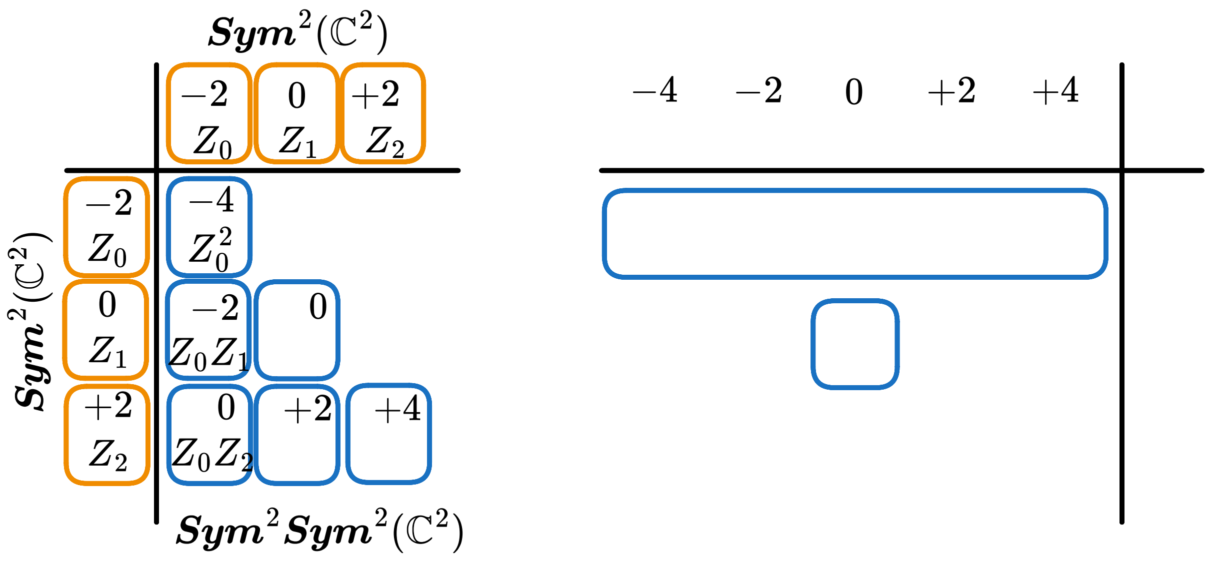
$\boldsymbol{Sym}^2(2) \cong_{\mathfrak{sl}(2,\mathbb{C})} 0 \oplus 4$

$\boldsymbol{Sym}^2\boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{\mathfrak{sl}(2,\mathbb{C})} \boldsymbol{Sym}^0(\mathbb{C}^2) \oplus \boldsymbol{Sym}^4(\mathbb{C}^2)$

using weights





decomposition as homogeneous polynomials

> Correspondences
>
> For any vector space W of dim $n+1$,
>
> $$\boldsymbol{Sym}^k W^* \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W)}_{}\}$$
> $$= \text{space of lines in } W$$
>
> Dually,
>
> $$\boldsymbol{Sym}^k W \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W^*)}_{}\}$$
> $$= \text{space of lines in } W^*$$
> $$= \text{space of hyperplanes in } W$$
>
> where the last correspondence is
>
> $$\begin{aligned} \text{lines in } W^* &\leftrightarrow \text{hyperplanes in } W \\ [\phi] &\leftrightarrow \ker\phi \end{aligned}$$
>
> Now because zero sets of polynomials are *hypersurfaces*(?)
>
> $$P(\boldsymbol{Sym}^k W) \leftrightarrow \{\text{hypersurfaces of deg } k \text{ in } P(W^*)\}$$
>
> so to derive results about $\boldsymbol{Sym}^k W$ we must work with $P(W^*)$.

We understand

- $$\boldsymbol{Sym}^2\boldsymbol{Sym}^2 V = \{\text{homg deg 2 polynomials on } \boldsymbol{P}(\boldsymbol{Sym}^2 V)^*\}$$
- for $V = \mathbb{C}^2 = \mathbb{C}\{x, y\}$, we have
$$\begin{aligned} \boldsymbol{Sym}^2\mathbb{C}\{x,y\} &\cong \mathbb{C}\{x^2, xy, y^2\} \cong \mathbb{C}\{Z_0, Z_1, Z_2\} \\ x^2 &\mapsto Z_0 \\ xy &\mapsto Z_1 \\ y^2 &\mapsto Z_2 \end{aligned}$$
- which gives
$$\boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2 = \underbrace{\mathbb{C}\left\{\begin{matrix} Z_0^2, Z_1^2, Z_2^2, \\ Z_0Z_1, Z_1Z_2, Z_2Z_0 \end{matrix}\right\}}_{\text{homog polynomials on } \boldsymbol{P}\mathbb{C}\{Z_0,Z_1,Z_2\} \cong \mathbb{C}P^2}$$
- thus using
  > - for $V = \mathbb{C}^2 = \mathbb{C}\{x, y\}$, we have
  > $$\begin{aligned} \boldsymbol{Sym}^2\mathbb{C}\{x,y\} &\cong \mathbb{C}\{x^2, xy, y^2\} \cong \mathbb{C}\{Z_0, Z_1, Z_2\} \\ x^2 &\mapsto Z_0 \\ xy &\mapsto Z_1 \\ y^2 &\mapsto Z_2 \end{aligned}$$

we have the evaluation map

$$\underbrace{\boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2} \to \underbrace{\boldsymbol{Sym}^4\mathbb{C}^2}$$

$$\mathbb{C}\left\{\begin{matrix} Z_0^2, Z_1^2, Z_2^2, \\ Z_0Z_1, Z_1Z_2, Z_2Z_0 \end{matrix}\right\} \to \mathbb{C}\left\{\begin{matrix} x^4 \\ x^3y, x^2y^2, xy^3 \\ y^4 \end{matrix}\right\}$$

$$\begin{aligned} Z_0^2 &\mapsto x^4 \\ Z_1^2 &\mapsto x^2y^2 \\ Z_2^2 &\mapsto y^4 \\ Z_0Z_1 &\mapsto x^3y \\ Z_1Z_2 &\mapsto xy^3 \\ Z_2Z_0 &\mapsto x^2y^2 \end{aligned}$$

- this map surjective because the chosen basis is contained in the image
- and it is easily seen

$$Z_2Z_0 - Z_1^2 \mapsto 0$$

  and this polynomial spans the kernel because the map is surjective and it goes from a dim 6 to a dim 5 space
- So we get an **exact sequence**

$$0 \to \mathbb{C}\{Z_2Z_0 - Z_1^2\} \hookrightarrow \boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2 \to \boldsymbol{Sym}^4\mathbb{C}^2 \to 0$$

- But what about the split $\boldsymbol{Sym}^4\mathbb{C}^2 \hookrightarrow \boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2$? We must define something like

$$\begin{aligned} \boldsymbol{Sym}^4\mathbb{C}^2 &\hookrightarrow \boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2 \\ x^4 &\mapsto Z_0^2 \\ x^3y &\mapsto Z_0Z_1 \\ x^2y^2 &\mapsto \frac{1}{2}(Z_1^2 + Z_0Z_2) \\ xy^3 &\mapsto Z_1Z_2 \\ y^4 &\mapsto Z_2^2 \end{aligned}$$

$\mathbb{C}P^2$ geometry behind the decomposition

For $n = 2$ we have the embedding

$$[x, y] \mapsto [x^2, xy, y^2]$$

whose image satisfies the equation

$$F(Z_0, Z_1, Z_2) := Z_0Z_2 - (Z_1)^2 = 0$$

in $\mathbb{C}P^2$ with coordinates

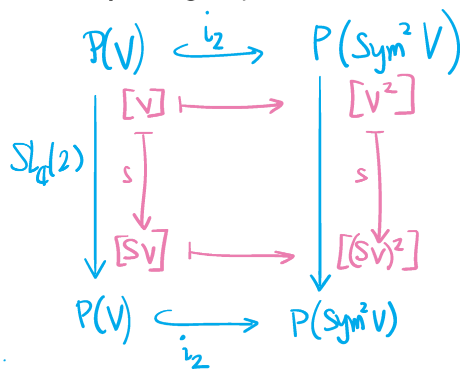
$$[Z_0, Z_1, Z_2] := [x^2, xy, y^2]$$

whose zero set is called the *plane conic* (rational normal curve of degree 2).

The action

$$\begin{aligned} SL_{\mathbb{C}}(2) \quad &\curvearrowright P\boldsymbol{Sym}^2V^* \cong \mathbb{C}P^2 \\ \hat{x}^k\hat{y}^{n-k} \quad &\overset{S}{\mapsto} (S\hat{x})^k(S\hat{y})^{n-k} \end{aligned}$$

is exactly the group of motions of $\mathbb{C}P^2$ that carry the plane conic c_2 onto itself



Thus the action

$$SL_{\mathbb{C}}(2) \curvearrowright \boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } P\ \underbrace{(W^*)}_{\boldsymbol{Sym}^2(V)}\ \}$$

must preserve $\mathbb{C}\{F\}$ spanned the polynomial for c_2.

The split

$$\boldsymbol{Sym}^4(V) \to$$

Hence

$$0 \to \underbrace{\mathbb{C}\{F\}}_{\boldsymbol{Sym}^0(V)} \hookrightarrow \boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \to \boldsymbol{Sym}^4(V) \to 0$$

is exact as $SL_{\mathbb{C}}(2)$-modules, that splits giving us an isomorphism

$$\boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^4(V) \oplus \boldsymbol{Sym}^0(V)$$

Current note has 0 direct children and 0 total descendants.

- stem
  - SL C 2
    - rep
      - FinVec$_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$
        - $\boldsymbol{Sym}^2\boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^0(\mathbb{C}^2) \oplus \boldsymbol{Sym}^4(\mathbb{C}^2)$

And it has 2 siblings.

- stem
  - SL C 2
    - rep
      - FinVec$_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$
        - $2 \otimes 3 = 5 \oplus 3 \oplus 1$ for FinVec$_{\mathbb{C}}$ irreps of $\mathfrak{sl}(2, \mathbb{C})$

- $\boldsymbol{Sym}^2\boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^0(\mathbb{C}^2) \oplus \boldsymbol{Sym}^4(\mathbb{C}^2)$