Info

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Almost simple groups

Definition. A Lie group is **almost simple** if its Lie algebra is *simple*.

By

Let G be a connected semi-simple Lie group. Then G is **compact** $\iff$ the Killing form on $\mathrm{Lie}(G)$ is **negative-definite**.

every connected Lie group corresponding to a Almost simple group with negative-definite Killing form is **compact**.

Almost simple algebra with negative-definite Killing form, AKA compact Lie $\mathbb{R}$-algebras	compact simple connected Lie groups	Z
$\mathfrak{su}(2,\mathbb{C})$	$SU(2,\mathbb{C}) \longrightarrow SO(3,\mathbb{R})$	$\mathbb{Z}_2$

And *every* connected Lie group corresponding to a *almost simple algebra* whose Killing form is not negative-definite is **non-compact**.

Almost simple algebra whose Killing form is not negative-definite, AKA non-compact Lie $\mathbb{R}$-algebra	signature of Killing form	non-compact simple connected Lie groups	Z
$\mathfrak{sl}(2,\mathbb{R})$	$(1,2)$	$\widetilde{SL(2)}(\mathbb{R}) \longrightarrow SL(2)(\mathbb{R}) \longrightarrow PSL(2)(\mathbb{R}) \cong SO^+(1$	$\mathbb{Z}$
$\mathfrak{sl}(n,\mathbb{R})$ for $n \geq 3$			

Classification of simple Lie $\mathbb{R}$-algebras

(Classification of simple Lie $\mathbb{R}$-algebras into complex and non-complex) Let $\mathfrak{g}$ be a simple Lie $\mathbb{R}$-algebra and $\mathfrak{g} \otimes \mathbb{C}$ be its complexification. Then either it is **complex**:

$$\mathfrak{g} \simeq_{\mathsf{LieAlg}_{\mathbb{R}}} \mathfrak{s}$$

for some Lie $\mathbb{C}$-algebra $\mathfrak{s}$ (and $\mathfrak{g} \otimes \mathbb{C} \simeq_{\mathsf{LieAlg}_{\mathbb{C}}} \mathfrak{s} \oplus \mathfrak{s}$) ***or*** it is **non-complex**: $\mathfrak{g} \otimes \mathbb{C}$ is simple over $\mathbb{C}$.

	simple Lie $\mathbb{C}$-algebra	compact simple Lie $\mathbb{R}$-algebra (*compact real form*)	non-compact non-complex simple Lie $\mathbb{R}$-algebras (*non-compact real forms*)
$A_n, n \geq 1$	$\mathfrak{sl}(n+1,\mathbb{C})$	$\mathfrak{su}(n+1,\mathbb{C})$	$\mathfrak{sl}(n+1,\mathbb{R})$
			$\mathfrak{su}(p,q)$
			$\mathfrak{sl}(m,\mathbb{H})$
$B_n, n \geq 2$	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n+1,\mathbb{R})$	$\mathfrak{so}(p,q,\mathbb{R})$
$C_n, n \geq 3$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{usp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R})$
			$\mathfrak{so}(p,2n-p,\mathbb{R})$
$D_n, n \geq 4$	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2n,\mathbb{R})$	$\mathfrak{so}(p,q,\mathbb{R})$
			$\mathfrak{so}(2n,\mathbb{H})$
E_6			
E_7			
E_8			
F_4			
G_2			

Theorem 6.105 (classification). Up to isomorphism every simple real Lie algebra is in the following list, and everything in the list is a simple real Lie algebra:

(a) the Lie algebra $\mathfrak{g}^{\mathbb{R}}$, where $\mathfrak{g}$ is complex simple of type A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4, or G_2,
(b) the compact real form of any $\mathfrak{g}$ as in (a),
(c) the classical matrix algebras

$$\begin{array}{lrl}
\mathfrak{su}(p,q) & \text{with} & p \geq q > 0,\ \ p+q \geq 2 \\
\mathfrak{so}(p,q) & \text{with} & p > q > 0,\ \ p+q \text{ odd},\ \ p+q \geq 5 \\
 & \text{or with} & p \geq q > 0,\ \ p+q \text{ even},\ \ p+q \geq 8 \\
\mathfrak{sp}(p,q) & \text{with} & p \geq q > 0,\ \ p+q \geq 3 \\
\mathfrak{sp}(n,\mathbb{R}) & \text{with} & n \geq 3 \\
\mathfrak{so}^*(2n) & \text{with} & n \geq 4 \\
\mathfrak{sl}(n,\mathbb{R}) & \text{with} & n \geq 3 \\
\mathfrak{sl}(n,\mathbb{H}) & \text{with} & n \geq 2,
\end{array}$$

(d) the 12 exceptional noncomplex noncompact simple Lie algebras given in Figures 6.2 and 6.3.

The only isomorphism among Lie algebras in the above list is $\mathfrak{so}^*(8) \cong \mathfrak{so}(6,2)$.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold group (Lie groups)
      - Almost simple groups

And it has 18 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold group (Lie groups)
      - Adjoint representation of a Lie group
      - Conjugation action of a Lie group on itself
      - Classification of Lie groups
      - Compact Lie groups
      - Connected Lie groups with discrete center
      - Connected Lie groups with non-surjective exponential map
      - Discrete subgroups of Lie groups
      - Smooth maps onto a Lie group
      - Lie groups with surjective exponential map
      - Maurer-Cartan form
      - Left-invariant Lagrangian on a Lie group
      - Lie algebra of a lie group
      - Bi-invariant metrics on Lie groups
      - Left invariant Riemannian metric on a Lie group
      - Semi-simple $\mathbb{R}$-Lie groups
      - Almost simple groups
      - Virtual subgroups of Lie groups
      - Lie group with dilations