Info

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Fuchsian groups of finite coarea

Let Γ be a Fuchsian group. Then Γ is geometrically finite and $\Lambda(\Gamma) = \partial_\infty \mathbb{R}\boldsymbol{H}^2$

$$\iff$$

$\mu\left({}^{\Gamma}\backslash^{\mathbb{R}\boldsymbol{H}^2}\right) < \infty$.

- ($\implies$) Let D be a fundamental polygon for Γ with finitely many sides and $\Lambda(\Gamma) = \partial_\infty \mathbb{R}\boldsymbol{H}^2$
  - If F has a free side then this side is not in $\Lambda(\Gamma)$. Thus F has no free sides.
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- ($\impliedby$)

conjugacy classes of maximal cyclic subgroups

Let Γ be a non-cocompact Fuchsian group of finite coarea. Then for any $\xi \in \partial_\infty D(\Gamma)$ there is a parabolic $\gamma \in \Gamma$ such that $\Gamma_\xi = \langle \gamma \rangle$ and for each $f \in \Gamma$ parabolic there exists $\xi \in \Gamma, g \in \Gamma$ such that $gfg^{-1} \in \Gamma_\xi$. Thus

$$\frac{\overbrace{\partial_\infty D(\Gamma)}^{\text{finite}}}{\Gamma} \leftrightarrow \frac{\left\{ \text{maximal cyclic} \left\langle \underbrace{\gamma}_{\text{parabolic}} \right\rangle \leq \Gamma \right\}}{\text{conjugacy}}$$
$$\xi \mapsto \Gamma_\xi$$

In particular, as $|\partial_\infty D(\Gamma)| < \infty$, we conclude that a Fuchsian group of finite coarea has finitely many conjugacy classes of maximal cyclic subgroups that is generated by a hyperbolic element.

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Let Γ be a Fuchsian group of finite coarea. Then there exists infinity many conjugacy classes of maximal cyclic subgroups of Γ that is generated by a hyperbolic element.

$$\left| \frac{\left\{ \text{maximal cyclic} \left\langle \underbrace{\gamma}_{\text{hyperbolic}} \right\rangle \leq \Gamma \right\}}{\text{conjugacy}} \right| = \infty$$

- As $\Gamma \leq PSL(2, \mathbb{R})$ is finitely generated by Selberg's lemma there exists a finite index, torsion-free subgroup $\Gamma_1 \leq \Gamma$.
  - Thus

$$\begin{tikzcd} \Gamma \arrow[r, phantom, "\overset{\text{free}}{\curvearrowright}"] & H^2 \arrow[d, "\text{covering}"] \\ & {}^{\Gamma}\backslash^{H^2} \end{tikzcd}$$

  - Let Γ is non-cocompact. Then $\Gamma_1 \cong F_n$ for some $n \geq 1$.
  - Let Γ is cocompact. Then $\Gamma_1 \cong \pi_1(T_g^2, x_0)$ for some $g \geq 2$.

Theorem 10.3.5. *Any non-elementary Fuchsian group contains infinitely many conjugacy classes of maximal hyperbolic cyclic subgroups.*

Proof. Suppose not, then there are hyperbolic elements $h_1, \ldots, h_t$ in G such that each hyperbolic element in G is conjugate to some power of some h_j. Let u and v be distinct limit points of G. By Theorem 5.3.8, there are

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hyperbolic elements $f_1, f_2, \ldots$ with distinct axes $A_1, A_2, \ldots$ such that A_n has end-points u_n and v_n where $u_n \to u$ and $v_n \to v$.

As each f_n is conjugate to some power of one of a finite number of the h_j, we may relabel and assume that $h_j = h_1$ for every n. Then

$$f_n = g_n (h_1)^{n_1} (g_n)^{-1},$$

say, and so the elements

$$q_n = g_n h_1 (g_n)^{-1}$$

have distinct axes A_n and the same translation length T as h_1. As A_n converges to the geodesic (u, v), this violates discreteness: explicitly, if $z \in (u, v)$, then

$$\begin{aligned} \sinh \tfrac{1}{2}\rho(z, q_n z) &= \sinh(\tfrac{1}{2}T) \cosh \rho(z, A_n) \\ &\to \sinh(\tfrac{1}{2}T) \end{aligned}$$

as $n \to +\infty$ yet the q_n are distinct. □

Current note has 0 direct children and 0 total descendants.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$

- Fuchsian groups $\Gamma <_{\mathsf{d}} PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
    - Fuchsian groups of finite coarea

And it has 17 siblings.

- stem
  - Objects stemming from $PSL(2)(\mathbb{R})$
    - Fuchsian groups $\Gamma <_{\mathsf{d}} PSL(2)(\mathbb{R})$ $\leftrightarrow$ oriented hyperbolic 2-manifold/orbifolds $\to$ Riemann surfaces with branching signature
      - Artihmetic Fuchsian groups and Artihmetic hyperbolic 2-orbifolds
      - Cocompact Fuchsian groups
      - Central simple algebra
      - Elementary Fuchsian groups
      - Finitely generated Fuchsian groups $\leftrightarrow$ geometrically finite Fuchsian groups
      - Fuchsian groups of finite coarea
      - Geodesic flow of Fuchsian groups and hyperbolic surfaces
      - Decomposition of hyperbolic surfaces
      - Infinitely generated Fuchsian groups
      - Limit set of Fuchsian groups
      - Marked hyperbolic surfaces
      - subst.R PSL 2 discrete subg.over Q-bar
      - Fuchsian Schottky groups
      - Spectrum of hyperbolic surfaces
      - Torsion-free Fuchsian groups
      - Cocompact torsion-free Fuchsian groups
      - Fuchsian triangle groups

THEOREM 4.2.5. *Suppose* Γ *has a non-compact Dirichlet region* $F=D_p(\Gamma)$ *with* $\mu(F)<\infty$. *Then*
(i) each vertex of F *at infinity is a parabolic fixed point for some* $T\in\Gamma$.
(ii) If ξ *is a fixed point of some parabolic element in* Γ, *then there exists* $T\in\Gamma$ s.t. $T(\xi)\in\partial_0(F)$.

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