Info

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# Riemannian submanifolds

**Proposition:**

**Lemma 2.11.** *Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold with or without boundary, $M$ is a smooth manifold with or without boundary, and $F\colon M \to \widetilde{M}$ is a smooth map. The smooth 2-tensor field $g = F^*\widetilde{g}$ is a Riemannian metric on $M$ if and only if $F$ is an immersion.*

► **Exercise 2.12.** Prove Lemma 2.11.

**Definition.** Isometric immersion/embedding

Let $(M, g)$ be a Riemannian manifold with/without boundary and

$$F : X \to M$$

is a smooth immersion of a smooth manifold $X$. Then

$$h = F^*g$$

is the **metric induced by** $F$ on $X$.
If $(X, h)$ were a Riemannian manifold, then an immsersion/embedding satisfying

$$F : (X, h) \to (M, g),\ F^*h = g$$

is called **isometric immersion/embedding**.

**Definition.** Riemannian submanifolds

Let

$$\iota : X \hookrightarrow M$$

is an immersed/embedded smooth submanifold (with/without boundary) of $(M, g)$ (without boundary). Then with the induced metric on $X$

$$h := \iota^* g$$

the Riemannian manifold $(X, h)$ is called a **Riemannian submanifold** (with/without boundary).

In other words, $h$ is just the metric $g$ restricted to $X$.

**Warning**

Even if the metric is pulled back, properties like *curvature* does not commute with the pullback.

**Definition.** Normal bundle and normal vector fields of submanifolds

Let

$$\iota : (X, h) \hookrightarrow (M, g)$$

is a Riemannian submanifold with/without boundary

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Then

$$\iota_* : T_pX \hookrightarrow T_pM$$

gives a split into an orthogonal direct sum

$$T_pM = T_pX \oplus N_pX$$

where the **normal vector space** $N_pX := T_pM^\perp$ contains the vectors **normal to** $X$, that is

$$v \in N_pX \iff \forall w \in T_pX\ g(v, w) = 0$$

The sub bundle

$$NX := \coprod_{p \in M} N_pX \leq TM$$

is called the **normal bundle of** $X$ whose sections are **normal vector fields along** $M$.

**Proposition 2.16 (The Normal Bundle).** *If $\widetilde{M}$ is a Riemannian $m$-manifold and $M \subseteq \widetilde{M}$ is an immersed or embedded $n$-dimensional submanifold with or without boundary, then $NM$ is a smooth rank-$(m-n)$ vector subbundle of the ambient tangent bundle $T\widetilde{M}|_M$. There are smooth bundle homomorphisms*

$$\pi^{\top}: T\widetilde{M}|_M \to TM, \qquad \pi^{\perp}: T\widetilde{M}|_M \to NM,$$

*called the **tangential** and **normal projections**, that for each $p \in M$ restrict to orthogonal projections from $T_p\widetilde{M}$ to $T_pM$ and $N_pM$, respectively.*

## boundary of Riemannian manifolds

Let $(M,g)$ is a Riemannian manifold with boundary

$$\iota : \partial M \hookrightarrow M$$

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Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Riemannian submanifolds

And it has 18 siblings.

- structures based on sets
  - manifolds
    - $\mathbb{R}$-manifold semi-Riemannian
      - Universal cover of semi-Riemannian manifolds
      - Curvature of a Riemannian manifold
      - Einstein manifolds
      - Exponential map from tangent bundle
      - Geodesics past conjugate points in a Riemannian manifold
      - Riemannian manifolds with ergodic geodesic flow
      - Isometries of a semi-Riemannian manifold
      - Laplacian on a Riemannian manifold
      - Parallel transport on a curve in a semi-Riemannian manifold
      - Conjugate points in a semi-Riemannian manifold
      - Quotients of Riemannian manifolds
      - Complete Riemannian manifolds
      - Riemannian connection
      - Riemannian submanifolds
      - Totally geodesic submanifolds of semi-Riemannian manifolds
      - Tangent bundle of a Riemannian manifold
      - Unit tangent bundle of a Riemannian manifold
      - Vector fields on Riemannian manifolds