Info

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Gromov hyperbolic spaces

Geodesics in a metric space

Definition. Arcs, rays and lines

A path

$$\gamma : I(\text{interval}) \subseteq \mathbb{R} \to M$$

is called	if
arc/segment joining $p, q \in M$	$I = [a, b]$ and $\gamma(0) = p, \gamma(1) = q$
ray originating at $p \in M$	$I = [0, \infty)$ and $\gamma(0) = p$
line	$I = \mathbb{R}$

Definition. Geodesics in a metric space

Let (M, d) be a metric space. A path

$$\gamma : I(\text{interval}) \subseteq \mathbb{R} \to M$$

is called	if
minimal geodesic	$$\forall t_1, t_2 \in I, d(\gamma(t_1), \gamma(t_2)) = \lvert t_1 - t_2 \rvert$$
unit speed (local) geodesic (locally (in time) minimal geodesic)	$\forall t \in I$ there exists a $\epsilon > 0$ such that $$d(\gamma(t_1), \gamma(t_2)) = \lvert t_1 - t_2 \rvert$$ for all $t_1, t_2 \in (t - \epsilon, t + \epsilon)$.

Therefore, we have

- *minimal geodesic segments* which are isometric embeddings $[0, L] \to M$
- *minimal geodesic rays* which are isometric embeddings $[0, \infty) \to M$
- *minimal geodesic lines* which are isometric embeddings $\mathbb{R} \to M$

Definition. Geodesic metric spaces

A metric space (M, d) is

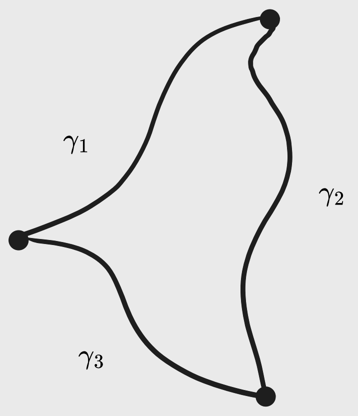
- a **geodesic metric space** or is **geodesically connected** if for every pair of points $p, q \in M$ there exists a *minimal geodesic* $\gamma : [a, b] \to M$ that joins them $\gamma(a) = p, \gamma(b) = q$
- **uniquely geodesic** if there is exactly *one* minimal geodesic joining x to y for each $x, y \in M$, denoted by $\overline{xy}$
- **geodesically complete** if every *minimal geodesic arc* $\gamma : [a, b] \to M$ can be extended to a *minimal geodesic line* $\tilde{\gamma} : \mathbb{R} \to M$.

Definition. Geodesic triangles

Let (X, d) be a metric space. A **geodesic triangle** is a triple

$$\gamma_1, \gamma_2, \gamma_3 : [0, L_1], [0, L_2], [0, L_3] \to X$$

of *geodesic segments*



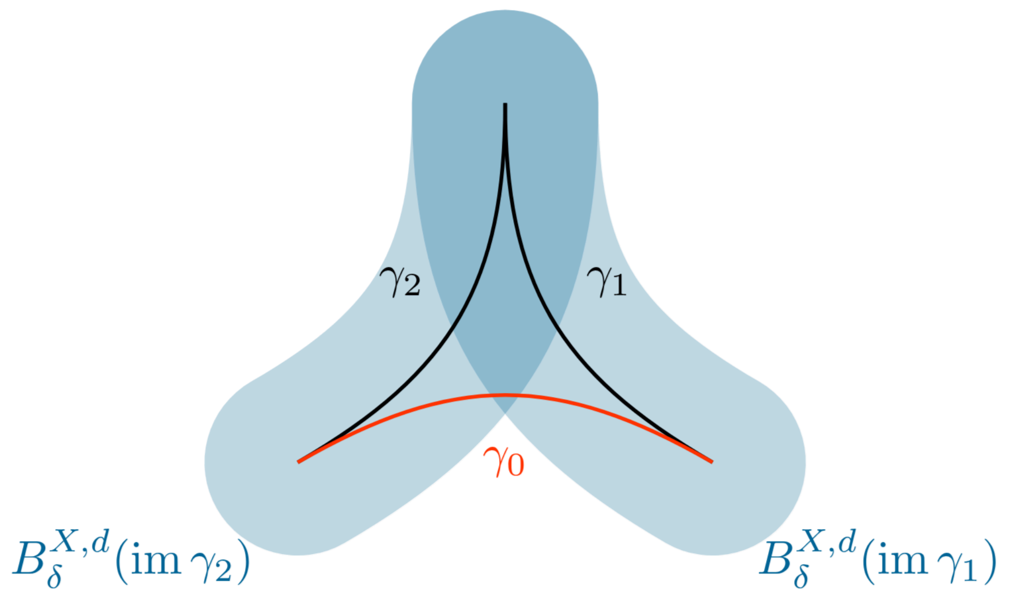
such that

$$\gamma_1(L_1) = \gamma_2(0)$$
$$\gamma_2(L_2) = \gamma_3(0)$$
$$\gamma_3(L_3) = \gamma_1(0)$$

Definition. Slim geodesic triangles and Gromov hyperbolic spaces

Let (X, d) be a metric space and $\delta \geq 0$.
A *geodesic triangle* $\gamma_1, \gamma_2, \gamma_3$ in (X, d) is **δ-slim** if each of its sides is contained in the δ-neigbourhood of union of the other two sides, that is,

$$\operatorname{im} \gamma_1 \subseteq B_\delta^{(X,d)}(\operatorname{im} \gamma_2 \cup \operatorname{im} \gamma_3)$$
$$\operatorname{im} \gamma_2 \subseteq B_\delta^{(X,d)}(\operatorname{im} \gamma_1 \cup \operatorname{im} \gamma_3)$$
$$\operatorname{im} \gamma_3 \subseteq B_\delta^{(X,d)}(\operatorname{im} \gamma_1 \cup \operatorname{im} \gamma_2)$$



A *geodesic metric space* (X, d) is **Gromov δ-hyperbolic** if every *geodesic triangle* in X is δ*-slim*.
The metric space (X, d) is **Gromov hyperbolic** if $\exists \delta > 0$ such that it is Gromov δ-hyperbolic.

Gromov quasi-hyperbolic spaces

Quasi-geodesics in a metric space

Definition. Arcs, rays and lines

A path

$$\gamma : I(\text{interval}) \subseteq \mathbb{R} \to M$$

is called	if
arc/segment joining $p, q \in M$	$I = [a, b]$ and $\gamma(0) = p, \gamma(1) = q$
ray originating at $p \in M$	$I = [0, \infty)$ and $\gamma(0) = p$
line	$I = \mathbb{R}$

Definition. Quasi-isometries between metric spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. A map

$$f : X \to Y$$

- is a (c, b)-**quasi-isometric embedding** if there are constants $c, b > 0$ such that $\forall x_1, x_2 \in X$

$$\frac{1}{c} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq c d_X(x_1, x_2) + b$$

- is **finite distance** from $f' : X \to Y$ if

$$\exists c > 0 : \forall x \in X, d_Y(f(x), f'(x)) \leq c$$

- has a **quasi-inverse** $g : Y \to X$ if $g \circ f$ has finite distance from Id_X and $f \circ g$ has finite distance from Id_Y
- is a **quasi-isometry** if it is a *quasi-isometric embedding* with a *quasi-inverse* $g : Y \to X$ which is also a quasi-isometry

Definition. Quasi-geodesics in a metric space

Let (M, d) be a metric space. A path

$$\gamma : I(\text{interval}) \subseteq \mathbb{R} \to M$$

is called a **quasi-geodesic** if it is a quasi-isometric embedding.

Therefore, we have

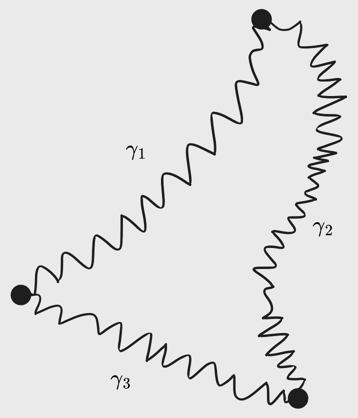
- *quasi-geodesic segments* which are quasi-isometric embeddings $[0, L] \to M$
- *quasi-geodesic rays* which are quasi-isometric embeddings $[0, \infty) \to M$
- *quasi-geodesic lines* which are quasi-isometric embeddings $\mathbb{R} \to M$

Definition. Quasi-geodesic triangles

Let (X, d) be a metric space. A **quasi-geodesic triangle** is a triple

$$\gamma_1, \gamma_2, \gamma_3 : [0, L_1], [0, L_2], [0, L_3] \to X$$

of *quasi-geodesic segments*



such that

$$\gamma_1(L_1) = \gamma_2(0)$$
$$\gamma_2(L_2) = \gamma_3(0)$$
$$\gamma_3(L_3) = \gamma_1(0)$$

Definition. Slim quasi-geodesic triangles and Gromov quasi-hyperbolic spaces

Let (X, d) be a metric space and $\delta \geq 0$.
A *quasi-geodesic triangle* $\gamma_1, \gamma_2, \gamma_3$ in (X, d) is **δ-slim** if each of its sides is contained in the δ-neigbourhood of union of the other two sides, that is,

$$\begin{aligned} \operatorname{im} \gamma_1 &\subseteq B_\delta^{(X,d)}(\operatorname{im} \gamma_2 \cup \operatorname{im} \gamma_3) \\ \operatorname{im} \gamma_2 &\subseteq B_\delta^{(X,d)}(\operatorname{im} \gamma_1 \cup \operatorname{im} \gamma_3) \\ \operatorname{im} \gamma_3 &\subseteq B_\delta^{(X,d)}(\operatorname{im} \gamma_1 \cup \operatorname{im} \gamma_2) \end{aligned}$$

A *quasi-geodesic metric space* (X, d) is **Gromov δ-quasi-hyperbolic** if every *quasi-geodesic triangle* in X is *δ-slim*.
The metric space (X, d) is **Gromov quasi-hyperbolic** if $\exists \delta > 0$ such that it is Gromov δ-quasi-hyperbolic.

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Metric spaces
    - Gromov hyperbolic spaces

And it has 11 siblings.

- structures based on sets
  - Metric spaces
    - Set of all metrics on a set
    - CAT(0)
    - Space of all compact metric spaces up to isometry
    - Y^X
    - Gromov hyperbolic spaces
    - Metric space isometries
    - Constructing more metrics
    - Metric spaces
    - Metric simplicial complexes
    - Metric subspaces
    - Uniformly discrete spaces of bounded geometry