Info

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n-times differentiable functions

functions n-times differentiable at a point

We generalize the preliminary result

> - From
>   > - Let $f:(a,b)\to\mathbb{R}$ be differentiable at $c\in(a,b)$.
>   >   - Define
>   >
>   > $$f_c^*(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{if } x\neq c \\ f'(c) & \text{if } x = c\end{cases}$$
>   >
>   >   > - f^* is continuous in (a,b).
>   >   - So we can write f as
>   >
>   > $$f(x)-f(c) = (x-c)f_c^*(x)$$
>   >
>   > for an unique continuous f_c^*.
>
>   we write
>
> $$f(x) = f(c) + (x-c)f_c^*(x)$$
>
>   as
>
> $$f(x) = f(x) + (x-c)f'(c) + (x-c)h_c(x)$$
>
>   where
>
> $$\begin{aligned} h_c(x) &:= f_c^*(x) - f(x) \\ \implies \lim_{x\to c} h_c(x) &= 0 \end{aligned}$$

for functions that are n-times differentiable at a point

> **(Peano form of Taylor's theorem)** Let Let $f:(a,b)\to\mathbb{R}$ and a positive integer $n\in\mathbb{Z}_{>0}$ such that f is n-times differentiable at $c\in(a,b)$. Then there exists a function $h_c:(a,b)\to\mathbb{R}$ such that
>
> $$f(x) = \sum_{k=0}^{n}\left(\frac{(x-c)^k}{k!}f^{(k)}(c)\right) + (x-c)^n h_c(x)$$
>
> and
>
> $$\lim_{x\to c} h_c(x) = 0$$

functions n-times differentiable on an interval

By using the *mean value theorem*

> **Corollary of mean value theorem:** If
>
> $$f:(a,b)\to\mathbb{R}$$
>
> is differentiable, then for any proper subinterval $(x_1,x_2)\subset(a,b)$ by mean value theorem we have
>
> $$\exists x\in(x_1,x_2) : f(x_2)-f(x_1) = (x_2-x_1)f'(x)$$

repeatedly, we obtain

> **(Mean value form of Taylor's theorem)** Let $f:[a,b]\to\mathbb{R}$ and a positive integer $n\in\mathbb{Z}_{>0}$ with $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). Then there exists $c\in(\alpha,x)\subseteq[a,b]$ such that
>
> $$f(x) = \sum_{k=0}^{n-1}\left(\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k\right) + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n$$
>
> (where the c depends on x).

repeated use of mean value theorem

$$P(x) := \sum_{k=0}^{n-1}\left(\frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k\right)$$

For a fixed interval $(\alpha,\beta)\subseteq[a,b]$, let M be the number defined by

$$f(x) = P(x) + M(\beta-\alpha)^n$$

and put

$$g(t) := f(t) - P(t) - M(t-\alpha)^n \text{ on } t\in(a,b)$$

Then

$$\begin{aligned} g^{(k)}(t) &= f^{(k)}(t) - P^{(k)}(t) - k!\, M(t-\alpha)^{n-k} \\ g^{(n)}(t) &= f^{(n)}(t) - n!M \text{ on } t\in(a,b) \end{aligned}$$

and for $0\le k\le n-1$ we have

$$P^{(k)}(\alpha) = f^{(k)}(\alpha) \implies g^{(k)}(\alpha) = 0$$

Also,

$$g(\beta) = 0$$

from definition.

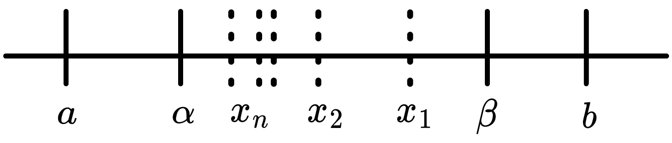
- Hence, as $g(\alpha) = 0 = g(\beta)$, by mean value theorem

$$\exists x_1 \in (\alpha, \beta) : g'(x_1) = 0$$

- Since $g'(\alpha) = 0 = g'(x_1)$, by mean value theorem

$$\exists x_2 \in (\alpha, x_1) : g'(x_2) = 0$$

- After n such steps



  we arrive at the conclusion that

$$\exists x_n \in (\alpha, x_{n-1}) \subset (\alpha, \beta) : g'(x_n) = 0$$

  Hence, there is a $c \in (\alpha, \beta)$ such that

$$f^{(n)}(c) = n!M$$

if first $n-1$ derivatives are zero at a point

If

$$f^{(k)}(\alpha) = 0$$

then

$$\begin{aligned} \exists c \in (\alpha, x) : f(x) &= \sum_{k=0}^{n-1} \left(\cancelto{0}{\frac{f^{(k)}(\alpha)}{k!}} (x-\alpha)^k \right) + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n \\ &= \frac{f^{(n)}(c)}{n!}(x-\alpha)^n \end{aligned}$$

if n-th derivative is continuous and non-zero at a point

- Let

$$\begin{aligned} f &\in \mathcal{C}^n[a,b] \\ \exists x_0 \in [a,b] &: f^{(n)}(x_0) < 0 \end{aligned}$$

> - Then by continuity of $f^{(n)}$, for any $\epsilon > 0$ we have a $\delta > 0$ such that
>
> $$|f^{(n)}(x) - f^{(n)}(x_0)| \leq \epsilon$$

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Current note has 0 direct children and 0 total descendants.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
    - mapping
      - Mean value property of functions
        - Taylor's theorem

And it has 1 siblings.

- stem
  - subobjects of and functions on $\mathbb{R}^n, S^n, \mathbb{C}^n$ and its quotients $\Gamma \backslash \mathbb{R}^n, \Gamma \backslash \mathbb{C}^n$
    - mapping
      - Mean value property of functions
        - Taylor's theorem