ⓘ Info

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$SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ is proper

via orbit map being closed

> **Proposition:** The map
>
> $$\begin{aligned} SL(2)(\mathbb{R}) &\to H^2_{\mathrm{U}} \\ g &\mapsto g(i) \end{aligned}$$
>
> is proper. ($\iff$ closed with compact fibers).

- Let $A \subseteq H^2$ be a compact subset. Then consider a sequence $\{g_n\} \subseteq SL(2)(\mathbb{R})$ in the preimage, that is

$$\begin{aligned} g_n(i) &\in A \\ \implies \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \cdot (i) &\in A \\ \implies \frac{a_n i + d}{c_n i + d} &\in A \\ \implies \mathfrak{I}\left(\frac{a_n i + d}{c_n i + d}\right) &= \frac{1}{|c_n i + d_n|^2} \xcancel{|\mathfrak{I}(i)|}^{1} \in \mathfrak{I}(A) \subset_{\mathsf{c}} (0, \infty) \\ \implies 0 < m_1 \leq |c_n i + d_n| &\leq M_1 \\ \implies |a_n i + d_n| &\leq M_2 \, |c_n i + d_n| \leq M_2 M_1 \end{aligned}$$

Therefore, each sequence $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ is bounded $\implies$ the preimage of A is compact.

Now we prove the action is proper.

- Let $\{p_i\}, \{g_i(p_i)\} \subseteq H^2_{\mathrm{U}}$ be convergent sequences.
  - Under

    $$\begin{aligned} SL(2)(\mathbb{R}) &\to H^2_{\mathrm{U}} \\ A &\mapsto A(i) \end{aligned}$$

    pick representatives γ_i such that $\gamma_i(i) = p_i$.
  - As $\{p_i\}$ is in a compact set, its preimage is compact and $\{\gamma_i\}$ is in the preimage. Thus $\{\gamma_i\}$ has a convergent subsequence.
  - Similarly $g_i \gamma_i(i) = g_i(p_i)$ is in a compact set, $\{g_i \gamma_i\}$ also has a convergent subsequence.
  - As inverse on $SL(2)(\mathbb{R})$ is a homeomorphism, γ_i^{-1} also has a convergent subsequence.
  - We observe

    $$g_i = (g_i \gamma_i)(\gamma_i)^{-1}$$

    thus has a convergent subsequence.
  - Thus by

    > **Proposition:** Let M be a topological manifold and let $G \curvearrowright M$ acts continuously. Then the following are equivalent
    >
    > 1. the action is proper
    >
    >   > Definition. A smooth Lie group action $\theta : G \curvearrowright M$ is **proper** if
    >   >
    >   > $$\begin{aligned} \pi_2 \times \theta : G \times M &\to M \times M \\ (g, m) &\mapsto (m, \theta_g(m)) \end{aligned}$$
    >   >
    >   > is *proper* as a continuous map.
    >
    > 2. $\{p_i\} \subseteq M, \{g_i\} \subseteq G$ are sequences such that both $\{p_i\}, \{g_i p_i\}$ converge $\implies$ a subsequence of $\{g_i\}$ converges
    > 3. for every compact $K \subseteq M$ the set
    >
    > $$\begin{aligned} G_K &:= \{g \in G | g(K) \cap K \neq \emptyset\} \\ &:= \pi_{G \times M \to G}((\pi_2 \times \theta)^{-1}(K \times K)) \end{aligned}$$
    >
    >   is compact.

    the action is proper.

via hyperbolic distance

Current note has 0 direct children and 0 total descendants.

- squishy
  - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$
    - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ is proper

And it has 1 siblings.

- squishy
  - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$
    - $SL(2)(\mathbb{R}) \curvearrowright H^2_{\mathrm{U}}$ is proper