Info

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Uniformly continuous functions on a metric space

Definition. Uniformly continuous functions on a metric space

Let $f : X \to Y$ be a function between metric spaces, then f is uniformly continuous if $\forall \epsilon, \exists \delta$ such that

$$x_1, x_2 \in X, d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon$$

Uniform continuity $\implies$ continuity

Let X be a compact metric space. $f : X \to Y$ is continuous $\implies$ f is uniformly continuous.

- Suppose f is not uniformly continuous, then $\exists \epsilon_0 > 0$ such that $\forall \delta$ there are $a, b \in X$ with $d(a, b) < \delta$ but $d(f(a), f(b)) \geq \epsilon_0$
  - take $\delta = \frac{1}{n}$ then there are $a_n, b_n \in X$ with $d(a_n, b_n) < \delta$ but $d(f(a_n), f(b_n)) \geq \epsilon_0$
  - now (a_n) and (b_n) are sequences in X
  - by sequential compactness there is a sub-sequence $(a_{n_k})_{k\in\mathbb{N}}$ which converges to $p \in X$
    - because f is continuous at p, $f(a_{n_k})_{k\in\mathbb{N}}$ also converges to$f(p)$
  - then

$$d(b_{n_k}, p) \leq d(b_{n_k}, a_{n_k}) + d(a_{n_k}, p) < \frac{1}{n_k} + d(a_{n_k}, p)$$

    - so b_{n_k} also converges to p
  - but

$$d(f(b_{n_k}), f(p)) \geq d(f(b_{n_k}), f(p)) - d(f(a_k), f(p)) \geq \epsilon_0 - d(f(a_{n_k}), f(p))$$

  - but this is a contradiction to continuity of f at p

Current note has 0 direct children and 0 total descendants.

- structures based on sets
  - Metric spaces
    - Metric spaces
      - Function between metric spaces
        - Uniformly continuous functions on a metric space

And it has 6 siblings.

- structures based on sets
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        - Topological entropy of maps between metric spaces
        - Limit of a function between metric spaces
        - Sensitive dependence on initial conditions of dynamics of a map between metric space