Info

This note, that was found here, as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
was created on June 22, 2024 6:29:42 PM,
was last modified on October 16, 2024 2:35:34 PM, and
was exported on September 7, 2026 3:04:01 PM.

Binomial coefficients

Definition. Binomial coefficients

For $n, k \in \mathbb{N}$ with $n \geq k$ the binomial coefficient

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

counts number of k element subsets of a set with n elements. By the binomial expansion we have

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

- For $x = 1$, we have

  $$2^n = \sum_{r=0}^{n} \binom{n}{r}$$

  which just says number of elements of power set equals sum of r element subsets for all r
- Multiply by $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ on both sides,

  $$(1+x)^{n-1} =$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$
$$= \frac{n!}{k!(n-k)!} \cancelto{1}{\left(\frac{n-k}{n} + \frac{k}{n}\right)}$$

This means Pascal's triangle may be defined using this recursive formula

```python
def pascal(n: int) -> list:
    r = []

    for i in range(1, n+1):
        row = [c := 1]
        for j in range(1, i):
            row.append(c := c * (i - j) // j)
        r.append(" ".join(map(str, row)))

    return r

if p := pascal(23):
    w = len(p[-1])
    for row in p:
        print(row.center(w))
```

$$\binom{2^m}{k} = \frac{2^m!}{k!(2^m-k)!} = \frac{2^m(2^m-1)\ldots(2^m-k+1)}{k!}?$$

- $$(1-x)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x^r$$

  - For $x = 1$, we have

    $$0 = \sum_{i=1}^{n} (-1)^r \binom{n}{r}$$

  - multiply by $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ on both sides

    $$(1-x)^{n-1} = \left(\sum_{r=0}^{n} (-1)^r \binom{n}{r} x^r\right)(1 + x + x^2 + \ldots)$$
    $$\sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} x^r = \sum_{r=0}^{n} \left(\sum_{i=0}^{r} (-1)^i \binom{n}{i}\right) x^r$$

    this means

    $$1 \leq r < n \implies (-1)^r \binom{n-1}{r} = \sum_{i=1}^{r} (-1)^i \binom{n}{i}$$

    [1]

- $$\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} = (-1)^{m-1}\binom{n-1}{m-1}$$
- $$\begin{aligned} n\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} &= (-1)^{m-1}\frac{m(n!)}{(m)!(n-m)!} \\ &= (-1)^{m-1}m\binom{n}{m} \end{aligned}$$

[2]

> Suppose $(a_n)_{n\in\mathbb{N}}$ is such that
>
> $$a_n + ua_{n-1} + va_{n-2} = 0$$
>
> such that
>
> $$\lambda^2 + u\lambda + v = 0$$
>
> has two distinct roots r_1, r_2. Then
>
> $$a_n = \alpha r_1^n + \beta r_2^n$$

$$a_k := \binom{n-k}{k}$$

$$\sum_{n\in\mathbb{N}}\sum_{r=0}^{n}\binom{n}{r}x^r t^n = \sum_{n\in\mathbb{N}}(1-x)^n t^n = \frac{1}{1-t(1-x)}$$

Current note has 0 direct children and 0 total descendants.

- stretchy
  - seq
    - Binomial coefficients

And it has 3 siblings.

- stretchy
  - seq
    - Binomial coefficients
    - Fibonacci sequence
    - Harmonic series

---

1. $$\begin{aligned} \sum_{k=0}^{D}(-1)^k\binom{n}{k} &= \sum_{k=0}^{D}(-1)^k \overbrace{\oint_{0<|z|=a<1}\frac{(1+z)^n}{z^{k+1}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}}^{=\binom{n}{k}} \\ &= \oint_{0<|z|=a<1}\frac{(1+z)^n}{z}\sum_{k=0}^{D}\left(-\frac{1}{z}\right)^k\frac{\mathrm{d}z}{2\pi\mathrm{i}} \\ &= \oint_{0<|z|=a<1}\frac{(1+z)^n}{z}\frac{(-1/z)^D+z}{1+z}\frac{\mathrm{d}z}{2\pi\mathrm{i}} \\ &= (-1)^D\underbrace{\oint_{|z|=a<1}\frac{(1+z)^{n-1}}{z^{D+1}}\frac{\mathrm{d}z}{2\pi\mathrm{i}}}_{=\binom{n-1}{D}} + \underbrace{\oint_{0<|z|=a<1}(1+z)^{n-1}\frac{\mathrm{d}z}{2\pi\mathrm{i}}}_{=0} \\ &= \boxed{(-1)^D\binom{n-1}{D}} \end{aligned}$$ ↩
2. [wiki.math.wisc.edu/images/Putnam_Binomial2020.pdf] (https://wiki.math.wisc.edu/images/Putnam_Binomial2020.pdf ↩