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Integral of holomorphic differential forms

Definition. Integration of k-forms on $\mathcal{C}^1$-singular chains on $U \subseteq \mathbb{C}$

- Integration of a *0-form* on a 0-chain are just sums of values
- Integration of a *1-form* $f\mathrm{d}z + g\mathrm{d}\bar{z}$ on a 1-chain is defined using integration on curves and extended R-linearly

$$\int_{\sum_{i=1}^{n} \alpha_i \gamma_i} f\mathrm{d}z + g\mathrm{d}\bar{z} = \sum_{i=1}^{n} \alpha_i \int_{\gamma_i} f\mathrm{d}z + g\mathrm{d}\bar{z}$$

- We may integrate *2-form* on a 2-chain B by

$$\int_B f\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

The holomorphic differential complex looks like

$$\begin{array}{ccccc} \mathcal{C}^{\mathcal{H}}(U,\mathbb{C}) \leq \Omega^0(U,\mathbb{C}) & \to & \Omega^{1,0}(U,\mathbb{C}) & \to & 0 \\ \phi & \overset{\mathrm{d}}{\mapsto} & \mathrm{d}\phi = \frac{\partial \phi}{\partial z}\mathrm{d}z + \cancelto{0}{\frac{\partial \phi}{\partial \bar{z}}}\mathrm{d}\bar{z} & & \\ & & f\mathrm{d}z & \overset{\mathrm{d}}{\mapsto} & 0 \end{array}$$

Thus for the holomorphic case, we restrict to *holomorphic 1-forms*

$$f\mathrm{d}z$$

where f is a holomorphic function and 2-forms I'm calling *holomorphic vector area form?*

$$f\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

where f is again a holomorphic function - which are *not* actually holomorphic 2-forms, because the only holomorphic 2-form is 0.

Forms like

$$|f|\mathrm{d}x \wedge \mathrm{d}y$$

also maybe?

Cauchy integral formula for holomorphic functions

holomorphic

(Cauchy integral formula) Let f be holomorphic in a disk around $z \in \mathbb{C}$ then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{S^1_r} \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w$$

- By the Cauchy integral formula,

$$\begin{aligned} \frac{\left|f^{(n)}(0)\right|}{n!} &\leq \frac{1}{2\pi} \frac{\|f|_{rS^1}\|_\infty}{r^{n+1}} (2\pi r) \\ &= \frac{\|f|_{rS^1}\|_\infty}{r^n} \end{aligned}$$

at the point of singularity

holomorphic forms $f(z)\mathrm{d}z$

The form

$$f(z)\mathrm{d}z$$

is closed *if and only if* f is holomorphic.

It is exact *if and only if* its integral is 0 identically.

(Conservative iff exact) Let $U \subseteq \mathbb{C}$ be any open set and $f : U \to \mathbb{C}$ be holomorphic. Then for all piecewise $\mathcal{C}^1$ closed curves γ on U,

$$\int_\gamma f(z)\mathrm{d}z = 0$$

if and only if **there exists a holomorphic function $F : U \to \mathbb{C}$ such that**

$$\frac{\partial F}{\partial z} = f \text{ on } U$$

The form

$$f(z)\mathrm{d}z$$

is exact implies its closed, implies its holomorphic.

(Morera's theorem) Any continuous function $f : U \to \mathbb{C}$ on open set $U \subseteq \mathbb{C}$ that satisfies

$$\int_\gamma f(z)\mathrm{d}z = 0$$

for every $\mathcal{C}^1$ closed curve γ in U must be holomorphic on U.

Of course the converse is not true for all holomorphic forms because

$$\int_{S^1} \frac{1}{z}\mathrm{d}z = 2\pi i$$

and the general statement is actually topological: depends on the topology of the domain of f.

(Cauchy-Goursat's theorem for convex open sets) If U is a *convex open set*, $f : U \to \mathbb{C}$ is holomorphic, then

$$\int_{\vec{abc}} f(z)\mathrm{d}z = 0$$

for every $a, b, c \in U$. This means

$$\int_\gamma f(z)\mathrm{d}z = 0$$

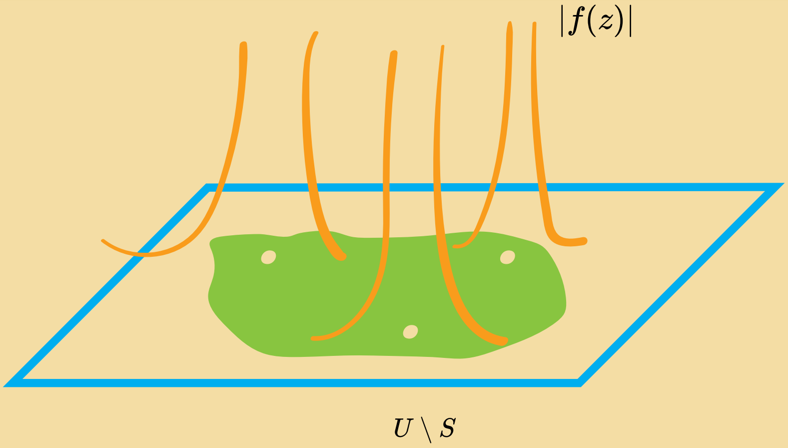
for any piecewise $\mathcal{C}^1$ closed curve γ on C.

Hence, holomorphic forms on convex sets are exact.

It is also true for simply connected sets, nothing but property of closed complex forms.

(Extended Cauchy-Goursat's theorem on convex sets) Let $U \subseteq \mathbb{C}$ be a convex open set and $S \subseteq U$ be a discrete subset. If a function $f : U \to \mathbb{C}$ holomorphic on $U \setminus S$ satisfies

$$s \in S \implies \lim_{z \to s} (z - s) f(z) = 0$$



then for any curve γ in $U \setminus S$ we have

$$\int_\gamma f(z)\mathrm{d}z = 0$$

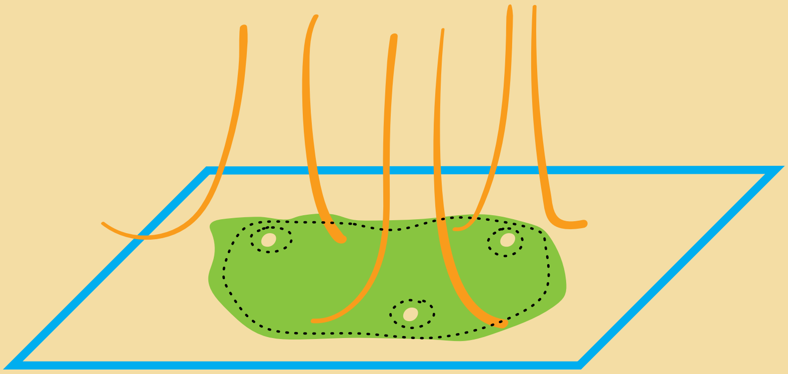
homology invariance of the integral

Because $f(z)$ is holomorphic if and only if $f(z)\mathrm{d}z$ is closed,

$$\int_B \cancelto{0}{\mathrm{d}(f(z)\mathrm{d}z)} = \int_{\partial B} f(z)\mathrm{d}z$$

so its integral over homologically trivial cycles is 0 identically.

(Cauchy-Goursat's theorem for any open set) If $f(z)$ is holomorphic in an open set U and *γ is a cycle homologous to zero in U*



then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

But this does not imply $f(z)$ is exact on U, obviously. Nothing *"analytic"* here, only closed complex form.

We can use that to derive at

(Cauchy integral with residues formula) Given a null homologous path in open $U \subseteq \mathbb{C}$ and a holomorphic function $f : U \setminus S \to \mathbb{C}$ where $A \subset U$ is a set without limit points in U we have

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\mathrm{d}z = \sum_{s\in S}\mathrm{Wind}(\gamma, s)\mathrm{Res}(f, s)$$

where $\mathrm{Wind}(\gamma, s) = 0$ for all but finitely many $s \in S$.

This implies if we have a meromorphic function with all residues equal to 0, then we have a anti-derivative for f in the domain of definition.

Does $\int_{\partial B} f(z)\mathrm{d}z = 0$ for all 2-chains B mean $f(z)\mathrm{d}z$ is closed?

homotopy invariance of the integral

[1]

weird complex forms using holomorphic functions

- Given a holomorphic 1-form $f\mathrm{d}z$ then

  $$\bar{f}\mathrm{d}\bar{z}$$

  is an *anti-holomorphic* 1-form and it has similar but complex conjugate properties to the holomorphic forms.
- Given a holomorphic f,

  $$f\mathrm{d}\bar{z}$$

  has very different properties,
  - conjugate is

    $$\bar{f}\mathrm{d}z$$

  - derivative is

    $$d(f\mathrm{d}\bar{z}) = \mathrm{d}f \wedge \mathrm{d}\bar{z} = \frac{\partial f}{\partial z}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

hence

$$\int_B \frac{\partial f}{\partial z} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{\partial B} f \mathrm{d}\bar{z}$$

Does Cauchy-Goursat's statement not hold for $f(z)\mathrm{d}\bar{z}$ with holomorphic $f(z)$?

Well,

$$\int_{S^1_{(R)}} z^k \mathrm{d}\bar{z} = R^k e^{ik\theta} \mathrm{d}(Re^{-i\theta}) = (-i)R^{k+1} \int e^{i(k-1)\theta} \mathrm{d}\theta = \begin{cases} & k > 1 \\ -2\pi R^2 i & k = 1 \\ 0 & k = 0 \\ & k < 0 \end{cases}$$

If f is holomorphic on a closed disk D of radius R then

$$\int_D \frac{\partial f}{\partial z} \mathrm{d}z \wedge \mathrm{d}\bar{z} = \int_{S^1_{(R)}} f \mathrm{d}\bar{z} = (-2\pi R^2 i) f'(0)$$

So

$$\int_D F \mathrm{d}x \wedge \mathrm{d}y = (2\pi R^2) F(0)$$

which is the harmonic function average property.

For a injective f holomorphic on open unit disk

$$\int_{\mathbb{D}} \left| \sum_{n \geq 0} b_n z^n \right|^2 \mathrm{d}x \wedge \mathrm{d}y = \pi \sum_{n \geq 0} |b_n|^2$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
            - Hol Equivalent descriptions of holomorphic functions
                - integral Integral of holomorphic differential forms

And it has 4 siblings.

- stamp stamp
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        - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
            - Hol Equivalent descriptions of holomorphic functions
                - Cauchy Cauchy integral formula for holomorphic functions
                - d Derivative of holomorphic functions
                - integral Integral of holomorphic differential forms
                - power series Convergent power series

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1. maths.tcd.ie/~zaitsev/topics-ca/wedhorn-Complex-Analysis.pdf#page=20.00 ↩