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is written (completely with human hands) by Rupadarshi Ray,
created on October 16, 2024 2:13:21 AM,
and was last modified on February 20, 2026 12:43:43 AM.

Equivalent descriptions of holomorphic functions

Definition. Holomorphic functions on $\mathbb{C}$

Given an open subset $U \subseteq \mathbb{C}$, a function

$$f : U \to \mathbb{C}$$

is called **holomorphic** if it satisfies any of the equivalent conditions

- $$\lim_{h\to 0, h\in\mathbb{C}\setminus\{0\}} \frac{f(z+h)-f(z)}{h}$$
  exists for all point $z \in U$
- $\iff$ f is *analytic* that is $\forall z_0 \in U, \exists \epsilon > 0$ such that
  $$f(z) = \sum_{n\geq 0} a_n (z-z_0)^n$$
  for all $z \in B(\epsilon, z)$
- $\iff$ f satisfies the Cauchy-Riemann equations
  $$\frac{\partial}{\partial \bar{z}} f = 0$$
- $\iff$ it satisfies Cauchy integral formula, that is, $f \in C^1(U)$, for all $z_0 \in U$, $\exists \epsilon > 0$
  $$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(\epsilon, z_0)} \frac{f(z)}{z - z_0}$$
- $\iff$ the complexified total derivative (whose Jacobian is)
  $\mathfrak{D}^{\mathbb{C}}_{z_0} f : T_{z_0}\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}$
  $$\begin{bmatrix} \frac{\partial}{\partial z} f & \frac{\partial}{\partial \bar{z}} f \\ \underbrace{\frac{\partial}{\partial z} \bar{f}}_{\frac{\partial}{\partial \bar{z}} f} & \underbrace{\frac{\partial}{\partial \bar{z}} \bar{f}}_{\frac{\partial}{\partial z} f} \end{bmatrix}$$

is diagonal $\iff$ it is a $\mathbb{C}$-linear map and then we have the complex total derivative (whose Jacobian is)

$$\left[\tfrac{\partial}{\partial z} f\right]$$

object	some derivative operators	equivalent to Cauchy-Riemann operator	Cauchy-Riemann equations is when the Cauchy-Riemann operator gives 0	integration	primitive	what does premitive satisfy
complex function $f = u + iv$	The *total derivative* $\mathfrak{D}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}u_x\\v_x$ which splits into $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$	$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\Big(($	Holomorphic		a holomorphic g such that $\frac{\partial g}{\partial z} = f$	
a (complex) differential $(1,0)$-form $f(z)\mathrm{d}z$		$\mathrm{d}(f(z)\mathrm{d}z) =$	closed differential form	$\int f(z)\mathrm{d}z$ $= \int (u+iv)$ $= \int (u\mathrm{d}x -$	an holomorphic g such that $\mathrm{d}g = f\mathrm{d}z$	
(real) differential 1-form $\omega = u\mathrm{d}x - v$ and its Hodge dual $\star\omega = v\mathrm{d}x +$		$\mathrm{d}\omega = (u_y +$ $\mathrm{d}\star\omega = (v_y -$	closed, coclosed differential form	$\int \omega + i \int$ $= \int (u\mathrm{d}x -$	a smooth ϕ such that $\mathrm{d}\phi = \omega$	$\mathrm{d}\star\mathrm{d}\phi = 0$

object	some derivative operators	equivalent to Cauchy-Riemann operator	Cauchy-Riemann equations is when the Cauchy-Riemann operator gives 0	integration	primitive	what does premitive satisfy
(real) vector field $\bar{f} = \begin{bmatrix} u \\ -v \end{bmatrix}$		$\text{curl}\bar{f} = -v_x$ $\text{div}\bar{f} = u_x$ −	*incompressible* (area preserving/ symplectic) , *solenoid* vector field	$\int \bar{f} \cdot (\gamma + i\gamma)$?	a smooth ϕ such that $\bar{f} = \text{grad}(\phi)$	$\Delta\phi = 0$
two real functions $u(x, y), v(x,$	Δ Δ $\text{grad}u \cdot \text{grad}$	$\text{grad}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	gradient of u rotated 90 degrees is the gradient of v - both are harmonic and conjugate to each other			
one real function $u(x, y)$			orthogonal real (harmonic?) functions $\|\text{grad}u\|_{z_0} =$ at some $z_0 \in \mathbb{C}$			

Current note has 4 direct children and 4 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Hol Equivalent descriptions of holomorphic functions

- Cauchy Cauchy integral formula for holomorphic functions
- d Derivative of holomorphic functions
- integral Integral of holomorphic differential forms
- power series Convergent power series

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$
      - space D closed $\mathscr{O}(\overline{D})$
      - space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
      - space D L2 $\mathscr{O} \cap L^2(D)$
      - space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - space Lp $\mathscr{O} \cap L^p$
      - space S1 $\mathscr{O}(S^1)$
      - zeros and singularities Zeros and singularities of holomorphic functions