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Meromorphic functions on the complex plane

Definition. Meromorphic function on the complex plane

A holomorphic function

$$f : z_0 + D_R^* \to \mathbb{C}$$

is **meromorphic at** z_0 if

$$|f(z)| \xrightarrow{z \to z_0} \infty$$

which means

$$\forall M > 0 \exists r \in (0, R) : |z - z_0| < r \implies |f(z)| > M$$

f is holomorphic on $z_0 + RD^*$	type of (isolated) singularities	the Laurent series $f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n$	image of f on $z_0 + \mathbb{D}_R^*$
	removable singularity, holomorphic at z_0	$c_n = 0$ for all $n < 0$	bounded
$(z - z_0)^k f(z)$ has a *removable singularity* for some $k > 0$ but f does not	pole, meromorphic at z_0	eventually $c_n = 0$	$\|f(z)\| \to \infty$ as $\|z\| \to 0$
$(z - z_0)^k f(z)$ does **not** have a removable singularity for any $k \in \mathbb{N}$	essential singularity	c_n does not become 0 eventually	image is **dense** on $\mathbb{C}$ (actually reaches almost all values on $\mathbb{C}$)

- If f is meromorphic at z_0 then

$$\frac{1}{f} : z_0 + D_r \to \mathbb{C}$$

is a non-constant holomorphic function with $z_0 \mapsto 0$.

  - This means by

    - If $f : U \to \mathbb{C}$ is a non-constant holomorphic function with

$$f(z) \equiv \sum_{n \geq k} a_n (z - z_0)^n \qquad \text{on } B(r, z_0)$$

    for $z_0 \in U$ where k is the smallest positive integer such that $a_k \neq 0$.

      - $\implies$

$$f(z) = (z - z_0)^k g(z) \qquad \text{on } B(r_1, z_0) \subseteq B(r, z_0)$$

      where g is a holomorphic function such that

$$g(z) :\equiv a_k + a_{k+1}(z - z_0) + \dots$$

      - As $g(z_0) = a_k \neq 0$, this means there is a smaller disk on which g is non-zero

$$g \neq 0 \qquad \text{on } B(r_2, z_0)$$

      - $\implies$ $f \neq 0$ on that same punchered disk around z_0
      - Thus zeroes of non-constant holomorphic functions on an open subset of $\mathbb{C}$ are isolated in $\mathbb{C}$, that is, the set of zeros $Z(f)$ is discrete in $\mathbb{C}$.

$$\frac{1}{f(z)} = (z - z_0)^k g(z)$$

  where g is holomorphic and $g \neq 0$ on a small disk near z_0 and k is a positive integer.

  - This mean $\frac{1}{g}$ is holomorphic on a small disk near z_0 and

$$f(z) = \frac{\frac{1}{g}(z)}{(z - z_0)^k}$$

  - Thus for any function f that is meromorphic at z_0, there is a holomorphic $h : z_0 + D_r \to \mathbb{C}$ and positive integer k called the **order of the pole at z_0** such that

$$f(z) = \frac{h(z)}{(z - z_0)^k} \text{ on } z_0 + D_r^*$$

  Moreover $h(z_0) \neq 0$.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - Mer Meromorphic functions on the complex plane

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$
      - space D closed $\mathscr{O}(\overline{D})$
      - space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
      - space D L2 $\mathscr{O} \cap L^2(D)$
      - space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - space Lp $\mathscr{O} \cap L^p$
      - space S1 $\mathscr{O}(S^1)$
      - zeros and singularities Zeros and singularities of holomorphic functions