Info

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created on March 17, 2024 10:23:35 AM,
and was last modified on April 27, 2024 8:37:26 AM.

Approximation of holomorphic functions on $\mathbb{C}$ by rational functions

any holomorphic function can be uniformly approximated by rational functions on a compact set with poles outside that compact set

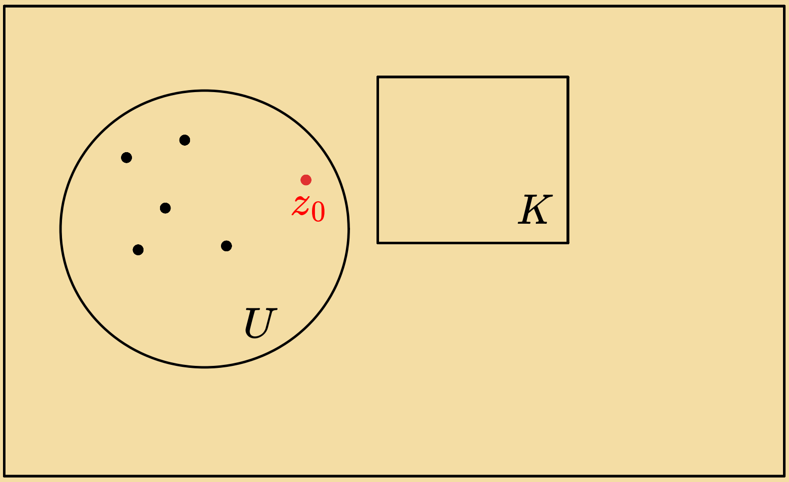
Let $f \in \mathcal{C}^{\mathcal{H}}(U)$ be a *holomorphic function* on an open subset $U \subseteq \mathbb{C}$, and let *$K \subseteq$ be compact*. Then for all $\epsilon > 0$ we have $b_1, \ldots, b_n$ in $\mathbb{C}$ and $a_1, \ldots, a_n$ in $\mathbb{C} \setminus K$ such that

$$\left| f(z) - \sum_{k=1}^{n} \frac{b_k}{z - a_k} \right| < \epsilon \text{ on } z \in K$$

poles of rational functions can be approximately pushed to any point

pushing poles around the plane

Let f be *a rational function with poles* on a connected open subset $U \subseteq \mathbb{C}$ and $K \subseteq \mathbb{C} \setminus U$ be a compact set.



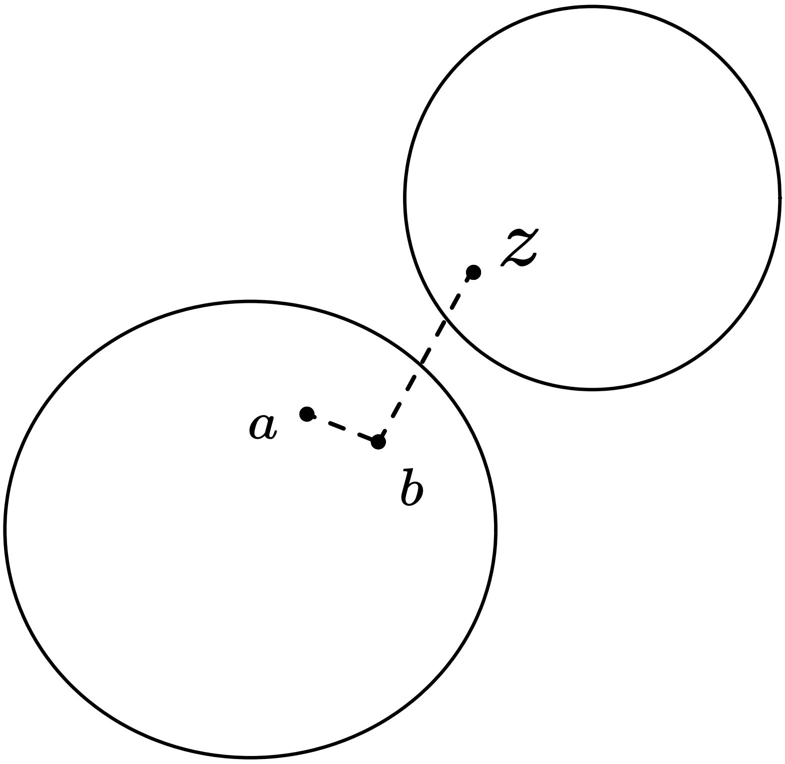
Then for all $z_0 \in U, \epsilon > 0$ there is a rational function of the form $P(z)/(z-z_0)^n$ such that

$$\left| f(z) - \frac{P(z)}{(z-z_0)^n} \right| < \epsilon \text{ on } z \in K$$

- On a compact $K \subset \mathbb{C}$, given $\epsilon > 0$, there is a rational function $\sum_{i=1}^n \frac{b_i}{z-a_i}$ where $a_i \notin K$ then

$$\left| f(z) - \sum_{i=1}^n \frac{b_i}{z-a_i} \right| < \epsilon$$

- Let U be connected open, $K \subseteq \mathbb{C} \setminus U$ compact.



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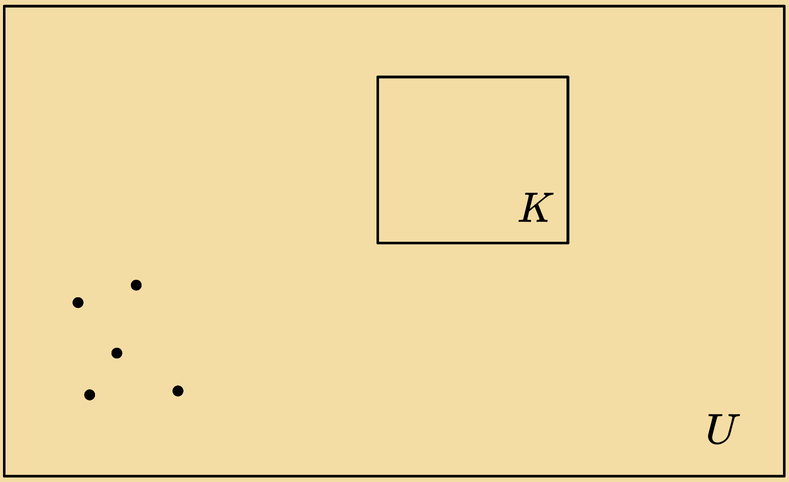
- $$|a-b| < |z-b| \implies \frac{1}{z-a} = \sum_{n \geq 0} \frac{(a-b)^n}{(z-b)^{n+1}}$$

- $$R_b := \{a \in U : \substack{\text{rat with pole at } a \\ \text{can be ufm approx by rat with pole at } b}\}$$
    - then R_b is open
    - and $R_b \subset U$ contains all its limit points
    - thus $R_b \subset U$ is open and closed, so $R_b = U$ as U is connected

pushing poles to infinity

Let f be *a rational function* with poles on a connected open subset $U \subseteq \mathbb{C}$ such that $\{z : |z| > R\} \subseteq U$ and $K \subseteq \mathbb{C} \setminus U$ be a compact set.

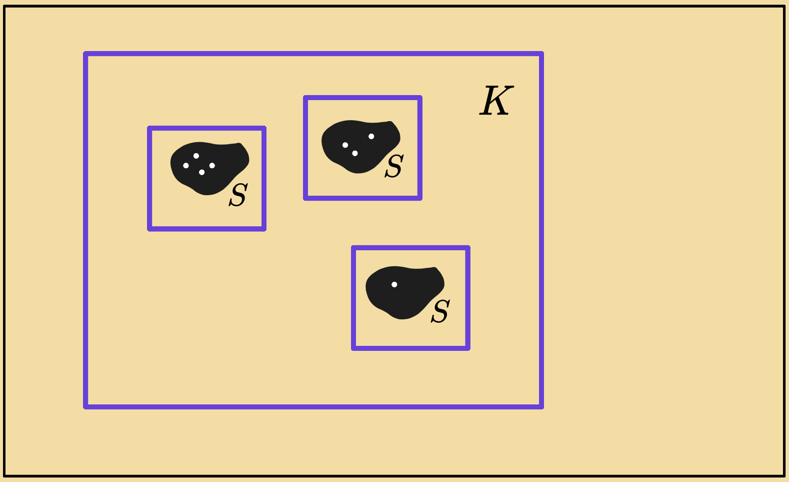


Then for all $\epsilon > 0$ there is a polynomial $P(z)$ such that

$$|f(z) - P(z)| < \epsilon \text{ on } z \in K$$

any holomorphic function can be uniformly approximated by rational functions on a compact set with poles *pushed anywhere* outside that compact set

(Runge's theorem) Let *$K \subseteq \mathbb{C}$ be compact*, *f be holomorphic* in some open interval containing K and $S \subseteq \mathbb{C} \setminus K$ which meets every connected component of $\mathbb{C} \setminus K$ except the one containing $\{z : |z| > R\}$ for some $R > 0$.



Then for every $\epsilon > 0$ there is a rational function

$$\sum_{k=1}^{n} P_k(z)/(z-a_k)^{n_k}$$

where P_k are polynomials, n_k are positive integers and $a_k \in S$ such that

$$\left| f(z) - \sum_{i=1}^{n} \frac{P_k(z)}{(z-a_k)^{n_k}} \right| < \epsilon \text{ on } z \in K$$

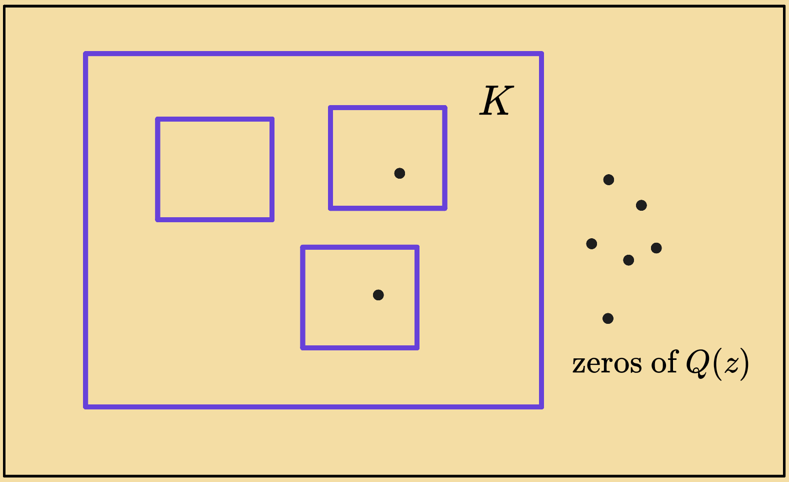
The rational function which approximates f has poles only in S. So, the statement also says if $\mathbb{C} \setminus K$ is connected, S may be empty, so the poles are at infinite/ approximation is by polynomials.

If $S \subseteq \mathbb{C} \setminus K$ meets every connected component of $\mathbb{C} \setminus K$, then

$$\forall \epsilon > 0 \left(\exists \frac{P(z)}{Q(z)}, z \in K \implies \left| f(z) - \frac{P(z)}{Q(z)} \right| < \epsilon \right)$$

and the zeroes of $Q(z)$ lies in S.

(Mergelyan's theorem) Let f be a continuous function on a compact set $K \subseteq \mathbb{C}$ which is *holomorphic on the interior of K*. Then for all $\epsilon > 0$, there is a rational function $P(z)/Q(z)$ where $Q(z)$ has zeros outside K



such that

$$\left|f(z) - \frac{P(z)}{Q(z)}\right| < \epsilon \text{ on } z \in K$$

We can also use the pole pushing argument as above to further control the zeros of $Q(z)$.

This result also generalises the *Weierstrass approximation theorem* in real analysis.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane

- Mer at infinity Holomorphic functions meromorphic at infinity
- modular Modular forms
- recons mer from stalk Reconstructing meromorphic functions from stalks
- reflection Extending holomorphic functions by reflections
- rotation symmetrizer Rotational symmetrization of holomorphic functions
- sheaf Sheaf of holomorphic functions on $\mathbb{C}$
- space $\mathscr{O}(U)$
- space C $\mathscr{O}(\mathbb{C})$
- space D $\mathscr{O}(D)$
- space D closed $\mathscr{O}(\overline{D})$
- space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
- space D L2 $\mathscr{O} \cap L^2(D)$
- space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
- space Lp $\mathscr{O} \cap L^p$
- space S1 $\mathscr{O}(S^1)$
- zeros and singularities Zeros and singularities of holomorphic functions