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Factorization of holomorphic functions on $\mathbb{C}$

infinite products

Definition. ***Non-zero*** **product of a sequence of non-zero complex numbers**

Given a sequence $(p_n) \subset \mathbb{C} \setminus \{0\}$ we have the partial products

$$p_0, p_0 p_1, \dots, \prod_{n \le k} p_n, \dots$$

Then the **infinite product** is defined to be the limit of the sequence of partial products

$$\prod_{n \ge 0} p_n := \lim_{k \to \infty} \prod_{n \le k} p_n$$

only if the right side is not 0.

- Let $p_n \in \mathbb{C} \setminus \{0\}$ be such that

  $$\prod_{n \ge 0} p_n = P \neq 0$$

  then

  $$p_n = \frac{\prod_{n \le k} p_n}{\prod_{n \le k-1} p_n} \xrightarrow{k \to \infty} 1$$

  - Thus it is necessary

    $$p_n \xrightarrow{n \to \infty} 1$$

    for the infinite product of p_n to converge.

$\prod_{n \ge 1} p_n$ converges

$$\iff$$

$\sum_n \log(p_n)$ **converges.**

constructing an entire function with given zeros and multiplicities

We shall use *Weierstrass' primary factors* and their bound

bound on $1 - E_n$

Proposition: The functions

Definition. Weirstrass' primary factors

$$
\begin{aligned}
E_0(z) &:= 1 - z \\
E_n(z) &:= (1 - z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right)
\end{aligned}
$$

for $n \in \mathbb{Z}_{>0}$

satisfy

$$|1 - E_n(z)| \leq |z|^{n+1}$$

on the unit disk D for all n.

For $m = 0$

$$|1 - (1 - z)| = |z|^1$$

For $m \geq 1$ we have

$$E'_m(z) = -z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$

which means

$$(1 - E_m)'(z) = z^m \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\right)$$

.

- As

$$
\begin{aligned}
\frac{1 - E_m(z)}{z^{m+1}} &= \sum_{n \geq 0} a_n z^n \\
1 - E_m(z) &= \sum_{n \geq 0} a_n z^{n+m+1}
\end{aligned}
$$

we have

$$\begin{aligned}\sum_{n\geq 0}\frac{a_n}{n+m+1}z^{n+m} &= -z^m\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^m}{m}\right)\\ &= -z^m\sum_{n\geq 0}\frac{1}{n!}\left(z+\frac{z^2}{2}+\cdots+\frac{z^m}{m}\right)^n\end{aligned}$$

Thus

$$a_n \geq 0$$

- So

$$(1-E_m)(0)=0$$

and $1-E_m$ has a zero of order $m+1$ at $z=0$. Thus

$$\frac{1-E_m(z)}{z^{m+1}}$$

has a removable singularity on 0, so it extends to an entire function

$$\begin{aligned}\frac{1-E_m(z)}{z^{m+1}} &= \sum_{n\geq 0}a_n z^n\\ \left|\frac{1-E_m(z)}{z^{m+1}}\right| &\leq \sum_{n\geq 0}|a_n||z|^n \leq \sum_{n\geq 0}|a_n| \text{ on } D\\ &= \sum_{n\geq 0}a_n\\ &= \frac{1-E_m(1)}{1^{m+1}}=1\end{aligned}$$

- Therefore,

$$|z|\leq 1 \implies |1-E_m(z)|M=|z|^{m+1}$$

(Entire function with given zeros and multiplicities) For a sequence $(a_n)\subset\mathbb{C}$ such that

$$|a_n| \xrightarrow{n\to\infty} \infty$$

let m be the number of times 0 occurs in a_n. Then

$$z^m\prod_{a_n\neq 0}E_n\left(\frac{z}{a_n}\right):\mathbb{C}\to\mathbb{C}$$

converges to a *holomorphic function* (entire) that has zeroes precisely at $\{a_n\} \subset \mathbb{C}$ with multiplicity at a zero a equal to the cardinality $|\{n \mid a_n = a\}|$.

- On choosing ϕ_n suitably such that

$$z^m \prod_{a_n \neq 0} E_{\phi_n}\left(\frac{z}{a_n}\right)$$

on a disk of radius $R > 0$

$$\sum_{n>N} \left|1 - E_{\phi_n}\left(\frac{z}{a_n}\right)\right| \leq \sum_{n>N} \left|\frac{z}{a_n}\right|^{\phi_n+1} \leq \sum_{n>N} \left|\frac{R}{a_n}\right|^{\phi_n+1}$$

converges.

Weierstrass Factorization Theorem

Let f be an entire function, $f \not\equiv 0$, and let $k \geq 0$ be the order of the zero of f at 0. Let the remaining zeros of f be at $z_1, z_2, \ldots,$ where each z_n is repeated as often as its multiplicity. Then

$$f(z) = e^{g(z)} z^k \prod_n E_{m_n}(z/z_n)$$

for some entire function g and nonnegative integers m_n.

Proof. If f has finitely many zeros, the result is immediate, so assume that there are infinitely many z_n. Since $f \not\equiv 0, |z_n| \to \infty$. By (6.2.3) there is a sequence $\{m_n\}$ such that

$$h(z) = f(z)/[z^k \prod_{n=1}^{\infty} E_{m_n}(z/z_n)]$$

has a zero-free extension to an entire function, which we will persist in calling h. But now h has an analytic logarithm g on $\mathbb{C}$, hence $h(z) = e^{g(z)}$ and we have the desired representation. ♣

More generally, versions of (6.2.3) and its consequence (6.2.5) are available for any *proper* open subset of $\hat{\mathbb{C}}$. We begin with the generalization of (6.2.3).

Holomorphic function with prescribed zeros and their multiplicities

(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A \subseteq \Omega$ with no limit point in Ω. For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a.

Ash and Novinger - Complex Variables , p.144

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - factorization Factorization of holomorphic functions on $\mathbb{C}$

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$
      - space D closed $\mathscr{O}(\overline{D})$
      - space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
      - space D L2 $\mathscr{O} \cap L^2(D)$
      - space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - space Lp $\mathscr{O} \cap L^p$

- space S1 $\mathscr{O}(S^1)$
- zeros and singularities Zeros and singularities of holomorphic functions