

### Info

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created on April 2, 2024 5:11:08 PM,  
and was last modified on April 13, 2026 8:07:28 PM.

## Global holomorphic functions

### Quote

**Riemann:** What is the *domain of definition* of an holomorphic function?

Let the union of all germs of holomorphic functions ("*Etale space*") be

$$\mathcal{O} := \left\{ (a, f) : a \in \mathbb{C}, f(z) = \sum_{n \geq 0} c_n (z - a)^n \right\}$$

and we want a topology on this such that

$$\begin{aligned} \mathcal{O} &\rightarrow \mathbb{C} \\ (a, f) &\mapsto a \end{aligned}$$

is a local homeomorphism.

We define

$$\begin{aligned} s_f : a + r_a D &\rightarrow \mathcal{O} \\ b &\mapsto (b, T_b f) \end{aligned}$$

where given a  $(a, f) \in \mathcal{O}$  we have

$$(T_b f)(z) := \sum_{n \geq 0} f^n(b) \frac{z - b}{n!}$$

Then the images are

$$s_f(a + r_a D) = \{(b, g) \mid\}$$

### Proposition:

$$\{s_f(a + r_a D) : (a, f) \in \mathcal{O}\}$$

becomes a sub-basis for a topology on  $\mathcal{O}$ .

Now

$\mathcal{O}$

is the space of all global holomorphic functions with (domain points from  $\mathbb{C}$  and valued in  $\mathbb{C}$ ).

On this topological space we have two maps

- the projection  $\pi$  onto  $\mathbb{C}$
- the *evaluation* of the function

$$\begin{array}{ccc}
 & \mathcal{O} & \xrightarrow{\text{ev}} \mathbb{C} \\
 & (a, f) & \xrightarrow{\quad} f(a) \\
 & \downarrow \pi & \\
 a + r_a D & \xrightarrow{\quad} & \mathbb{C} \\
 & & a
 \end{array}$$

**Definition 1.** We define the *derivative*  $d$ , a map  $d: \mathcal{O} \rightarrow \mathcal{O}$  as follows: If  $f_a \in \mathcal{O}_a$  and  $(U, f)$  is a representative of  $f_a$ , we define  $d(f_a) = df_a$  to be the germ at  $a$  of  $(U, f')$  where  $f' = df/dz$  is the derivative of  $f$ .

**Proposition 1.**  $d: \mathcal{O} \rightarrow \mathcal{O}$  is a covering map.

*Proof.* Let  $f_a \in \mathcal{O}$  and let  $(U, f)$  be a representative of  $f_a$ . Let  $D$  be a disc centered at  $a$  such that  $D \subset U$ .

Let  $F$  be a primitive of  $f$  on  $D$ , and let  $\mathcal{D} = N(D, f)$ . For any  $c \in \mathbb{C}$ , let  $U_c = N(D, F + c)$ . We claim that  $d^{-1}(\mathcal{D}) = \bigcup_{c \in \mathbb{C}} U_c$ .

To prove this, let  $z \in D$ ,  $g_z \in \mathcal{O}_z$  and  $dg_z = f_z$ . Let  $(W, g)$  be a representative of  $g_z$  where  $W$  is a connected neighborhood of  $z$ ,  $W \subset D$ . Then  $g' = f$  in a neighborhood of  $z$ , hence  $g' = f$  on  $W$  so that  $(d/dz)(g - F) = 0$  on  $W$ . Hence, for some  $c \in \mathbb{C}$ ,  $g = F + c$  on  $W$  and  $g_z \in U_c$ . Conversely, we trivially have  $dU_c = N(D, f) = \mathcal{D}$ .

We now check that  $d|_{U_c}$  is a homeomorphism onto  $\mathcal{D}$  for any  $c \in \mathbb{C}$ . We have only to check that  $d|_{U_c}$  is injective, which is obvious since  $d$  takes distinct elements of  $U_c$  to germs at different points of  $D$ . Since, finally, the  $U_c$  with distinct values of  $c$  are mutually disjoint,  $\mathcal{D}$  is evenly covered by  $d$ .

We shall now show that there is a close connection between integration along curves and the lifting of curves relative to the derivative  $d: \mathcal{O} \rightarrow \mathcal{O}$ .

## connected components

A connected component of  $\mathcal{O}$  is a "Riemann surface"  $S$ . There are two functions on it

$$S \rightarrow \mathbb{C}; (a, f) \mapsto a$$

$$S \rightarrow \mathbb{C}; (a, f) \mapsto f(a)$$

- a  $(0, f)$  whose connected component only contains disk?

## constructing global holomorphic functions

- [The complex logarithm](#)
- Radicals  $z^{1/k}$
- Inverse of a holomorphic function  $f : U \rightarrow \mathbb{C}$
- Primitive if a 1-form  $f(z)dz$  on  $U \subseteq \mathbb{C}$
- Radical  $\sqrt[k]{f}$  of a holomorphic function  $f : U \rightarrow \mathbb{C}$

## adding the missing points: meromorphic functions on $\mathbb{C}P^1$

The corresponding space for meromorphic functions on  $\mathbb{C}P^1$  is

$$\mathcal{M} := \left\{ (a, f) \mid f = \sum_{n=-N}^{\infty} c_n (z-a)^n, \limsup_n |c_n|^{1/n} < \infty \right\} \\ \cup \left\{ (\infty, f) \mid f(z) = \sum_{n=-N}^{\infty} c_n \frac{1}{z^n}, \limsup_n |c_n|^{1/n} < \infty \right\}$$

So

$$\mathcal{O} \subset \mathcal{M}$$

but its larger because

$$\left(0, \frac{1}{z}\right) \in \mathcal{M}$$

We extend the topology on  $\mathcal{O}$  to a topology on  $\mathcal{M}$ .

We have the extended maps

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\text{ev}} & \mathbb{C}P^1 \\ (a, f) & & f(a) \\ \pi \downarrow & & \\ \mathbb{C}P^1 & & \\ a & & \end{array}$$

where

**Proposition:**  $\pi$  is a local homeomorphism.

☀ (check for the points at infinity...)

## adding the missing points: finite ramifications

Consider  $rD^*$  and its inverse image in  $\mathcal{M}$ . There are many connected components of the pre-image. We consider a component  $E \subset \mathcal{M}$  such that

$$\pi : E^* \rightarrow rD^*$$

is **proper**.

These exist.

And now by

### Proposition:

11-9. Show that a proper local homeomorphism between connected, locally path-connected, and compactly generated Hausdorff spaces is a covering map.

11-10. Show that a covering map is proper if and only if it is finite-sheeted.

we know

$$\pi : E^* \rightarrow rD^*$$

is a finite, connected, covering map of the punctured disk!

They are classified by finite subgroups  $\pi_1(D^*) \cong \mathbb{Z}$ , and they are of the form:

$$D^* \rightarrow D^*, z \mapsto z^n$$

or

$$\mathcal{O}[z^{1/n}] \times_{\mathbb{C}^*} D^* = D_{1/n}^* =: \pi^{-1}(D^*) \xrightarrow{\pi} D^*$$

which gives

$$\begin{array}{ccccc} & & E^* & \longrightarrow & D^* \\ & \swarrow f & \uparrow & \nearrow & \\ \mathbb{C}P^1 & \longleftarrow & D_{1/n}^* & \longrightarrow & D_{1/n} \end{array}$$

$$\prod_i (w - f \circ \gamma^i) = w^n + a_1 w^{n-1} + \dots + a_n$$

then these  $a_i$  are holomorphic functions well defined on  $D^*$  which has a removable singularity at 0, so they extend onto  $D$ . Now by

## ☰ (Newton-Puiseux series)

[1]

we can solve for  $f$  on  $D_{1/n}$ .

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Now we add these points

$$(0, -)$$

...

and we construct

$$\overline{\mathcal{M}}$$

adds onto  $\mathcal{M}$  the "punctures" to the punctured disks in  $\mathcal{M}$  that occurs as finite covers of punctured disks in  $\mathbb{C}P^1$ .

The components of  $\overline{\mathcal{M}}$  solves the problem of finding the maximal domain of definition of a meromorphic function on  $\mathbb{C}P^1$ .

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    - [1Hol](#) Holomorphic functions on spaces over  $\mathbb{C}$  of dimension 1
      - [global](#) Global holomorphic functions

And it has 22 siblings.

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  - [Rf](#) subobjects of and functions on  $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$ 
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- [recons mer from stalk](#) Reconstructing meromorphic functions from stalks
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- [rotation symmetrizer](#) Rotational symmetrization of holomorphic functions
- [sheaf](#) Sheaf of holomorphic functions on  $\mathbb{C}$
- [space](#)  $\mathcal{O}(U)$
- [space C](#)  $\mathcal{O}(\mathbb{C})$
- [space D](#)  $\mathcal{O}(D)$
- [space D closed](#)  $\mathcal{O}(\overline{D})$
- [space D cnt bd](#)  $\mathcal{O}(D) \cap \mathcal{C}(\overline{D})$
- [space D L2](#)  $\mathcal{O} \cap L^2(D)$
- [space H](#)  $\mathcal{O}^p(H_{\mathbb{D}}^2)$
- [space Lp](#)  $\mathcal{O} \cap L^p$
- [space S1](#)  $\mathcal{O}(S^1)$
- [zeros and singularities](#) Zeros and singularities of holomorphic functions