Info

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Bounding derivative of holomorphic mappings

(Swartz inequality)

$$f \in \mathscr{O}(D, D), f(0) = 0 \implies \begin{cases} |f'(0)| \leq 1 \\ |f(z)| \leq |z| \end{cases}$$

Moreover, equality in *either* of these inequalities $\iff$ its a rotation $f(z) \equiv cz, |c| = 1$.

Consider a holomorphic mapping

$$f : D \to D$$

such that $f(0) = 0$.

- Then

$$f(z) \equiv zg(z)$$

  for a holomorphic $g : D \to \mathbb{C}$ giving us

$$f'(z) = g(z) + zg'(z)$$

- Applying

> Because modulus of non-constant holomorphic functions cannot attain its maximum in the interior of its domain, on the disk D_R of radius R, a holomorphic function that is continuous on the closed disk needs to have its maximum at the boundary $S^1_{(R)}$.

  for $r < 1$

$$g(z)\Big|_{D_r}$$

  we conclude its maximum should lie at some point z_r such that $|z_r| = r$.

- We see

$$f(z_r) \in D \implies |f(z_r)| \leq 1 \implies |g(z_r)| \leq \frac{1}{r}$$

  while the equality holds only if g is constant (note $r < 1$).
- Since, above is true for all $r < 1$ we see

$$|z| < 1 \implies |g(z)| \leq 1$$

  with equality holding $\iff$ g is constant.

Analogously,

Proposition: Consider a holomorphic function f that maps $f : B(R_0, z_0) \to B(M_0, f(z_0))$, that is

$$|f(z) - f(z_0)| < M_0 \qquad |z - z_0| < R_0$$

Then

$$|f(z) - f(z_0)| \leq \frac{M_0}{R_0}|z - z_0| \qquad |z - z_0| < R_0$$

By

(Swartz inequality)

$$f \in \mathscr{O}(D, D), f(0) = 0 \implies \begin{cases} |f'(0)| \leq 1 \\ |f(z)| \leq |z| \end{cases}$$

Moreover, equality in *either* of these inequalities $\iff$ its a rotation $f(z) \equiv cz, |c| = 1$.

on

$$g(z) := f(R_0 z) - f(z_0)$$

Cauchy's bounds on derivatives

(Cauchy's bounds on derivatives of holomorphic functions) Let $f \in \mathscr{O}(U)$. Then

$$\frac{\left|f^{(n)}(a)\right|}{n!} \leq \frac{\left\|f|_{B(r,a)}\right\|_\infty}{r^n}$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - bd derivative Bounding derivative of holomorphic mappings

And it has 9 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - bd derivative Bounding derivative of holomorphic mappings
        - branch points Branch points and ramification index of holomorphic functions
        - contains disks Disks in holomorphic images
        - fibers Fibers of holomorphic mappings
        - locally poly Non-constant holomorphic functions locally look like w^k and are open mappings
        - onto disk Biholomorphic mapping onto disk
        - pole Holomorphic functions on punchered disk
        - singularities are not isolated Holomorphic function whose singularities are not isolated
        - z2 at all rationals A holomorphic map branched over $\mathbb{Q}$