

Info

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Non-constant holomorphic functions locally look like w^k and are open mappings

☰ (Open mapping theorem for open subsets in \mathbb{C}) Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected). >

non-constant holomorphic function locally looks like w^k on a change of coordinates

Let us have a holomorphic function f on some open subset of \mathbb{C} containing z_0 .

Proposition: If $f(z_0) \neq 0$ and *any* positive integer k , we compose f with the function $z^{1/k}$ to obtain a function h such that

$$h^k = f \text{ on } V$$

for a smaller open V containing z_0 .

Proposition: If $f : U \rightarrow \mathbb{C}$ is a *non-constant* holomorphic function on an open subset $U \subseteq \mathbb{C}$,

- then around any $z_0 \in U$ there is a holomorphic $g(z)$

$$f(z) = f(z_0) + (z - z_0)^k g(z) \text{ around } z_0 \\ g(z_0) \neq 0$$

- Moreover there is a smaller neighborhood around z_0 and a holomorphic $h(z)$ such that

$$f(z) = f(0) + ((z - z_0)h(z))^k \text{ around } z_0$$

$$h(z_0) \neq 0$$

with further smaller neighborhood around z_0 so that $(z - z_0)h(z)$ is a biholomorphism to its (open) image.

- Now, we write

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \text{ on } z \in B_R(z_0)$$

- If f is **not constant**, there is some smallest $k \geq 1$ (**multiplicity of the zero**) such that $a_k \neq 0$, meaning

$$f(z) = a_0 + (z - z_0)^k g(z) \text{ on } z \in B_R(z_0)$$

$$g(z) = \sum_{n \geq 0} a_{n+k} (z - z_0)^n$$

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we find a smaller open V containing z_0 and a holomorphic h such that

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- Thus we have

$$f(z) = a_0 + \underbrace{((z - z_0)h(z))^k}_{h_1(z)} \text{ on } z \in V$$

with $h(z_0) \neq 0$, so

$$h_1(z) = (z - z_0)h(z)$$

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- As h_1 has a non-vanishing derivative at z_0 , so there must be **another** smaller open W containing z_0 where $h_1(W)$ is open and

$$h_1 : W \rightarrow h_1(W)$$

is a homeomorphism.

- We can choose W so that $h_1(W)$ is a disk centered at $h_1(z_0) = 0$.

Proposition:

$$f : W \rightarrow f(W) = a_0 + D$$

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is an **open** map.

- ☀ Given open set $A \subseteq W$ its image $h_1(A)$ under the homeomorphism h_1 is open set in a disk containing 0.
- Image of $h_1(A)$ under $z \mapsto z^k$ is open and translation $w \mapsto a_0 + w$ takes open sets to open sets.
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modulus of a non-constant holomorphic function

Let $f : U(\text{open}) \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant holomorphic function, then by

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$$\begin{aligned}h_1(z) &= (z - z_0)h(z) \\h_1'(z) &= h(z) + (z - z_0)h'(z) \\h_1'(z_0) &= h(z_0) \neq 0\end{aligned}$$

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- We can choose W so that $h_1(W)$ is a disk centered at $h_1(z_0) = 0$.

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- for all $z \in U$ there is a $u \in U$ such that

$$|f(u)| > |f(z)|$$

that means $|f|$ does not have a maximum at any point of U

- minimum can only be 0 because otherwise the modulus of the holomorphic

$$\frac{1}{f(z)}$$

attains a maximum.

- Thus, image of $|f|$ is *connected open* in $[0, \infty)$.

For two holomorphic functions f, g on a connected open set U we get $f - g$ on U which can either be constant or have no maximum modulus on U, \dots this probably tells us nothing.



holomorphic functions to closed sets must map to interior

Let

$$f : \Omega \rightarrow \bar{U}$$

is a *non-constant* holomorphic function from an open $\Omega \subseteq \mathbb{C}$ to a closed set \bar{U} with open interior U .

Then by

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we must have

$$f(\Omega) \subseteq U$$

Even, by (the statement of) [maximum modulus](#) we may conclude the same for function

$$f : \Omega \rightarrow B_R(0)$$

going to a ball centered at 0: if there were a point in the image which is in the boundary of the ball

$$p \in U, f(p) \in f(\Omega) \cap S_R^1$$

then p would be a maxima for $|f|$.

This has implications:

- Let

$$f_n : \Omega \rightarrow U$$

be a sequence of functions converging to f uniformly on compact subsets of Ω . By

☰ Let $f_n : U \rightarrow \mathbb{C}$ be holomorphic on open $U \subseteq \mathbb{C}$ and

$$f_n \xrightarrow{\text{uniformly as } n \rightarrow \infty} f \text{ on } K$$

for every $K \subset U$ compact then f is holomorphic in U .

the limit f is holomorphic. Then as $f_n(z) \in U$ the limit must be in \bar{U} , but because image must be open, so we conclude the limit is a holomorphic

$$f : \Omega \rightarrow U$$

- So we have

$$z \in \Omega \implies \lim_{n \rightarrow \infty} f_n(z) \in U$$

even though $f_n(z) \in U$ and U is open.

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 - [1Hol](#) Holomorphic functions on spaces over \mathbb{C} of dimension 1
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And it has 9 siblings.

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 - [pole](#) Holomorphic functions on punched disk

- singularities are not isolated Holomorphic function whose singularities are not isolated
- z^2 at all rationals A holomorphic map branched over \mathbb{Q}