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is written (completely with human hands) by Rupadarshi Ray,
created on February 23, 2025 12:21:37 PM,
and was last modified on June 12, 2026 10:57:27 PM.

Biholomorphic mapping onto disk

(Riemann mapping theorem) Let $\Omega \subset \mathbb{C}$ be a simply connected open, proper subset of $\mathbb{C}$. Then there exists a biholomorphism from Ω onto the unit disk

$$f : \Omega \to D$$

Moreover, given $a \in \Omega$ there is a unique biholomorphism

$$\begin{gathered} f : \Omega \to D \\ f(a) = 0, f'(a) > 0 \end{gathered}$$

uniqueness and maximality condition

Proposition: Let $\Omega \subseteq \mathbb{C}$ be a proper, simply connected subset of $\mathbb{C}$. Then

1. Given $a \in \Omega$ there is at most one biholomorphism

$$\begin{gathered} f : \Omega \cong D \\ f(a) = 0, f'(a) > 0 \end{gathered}$$

2. There exists an injective holomorphic map

$$f : \Omega \hookrightarrow D$$

3. Let $f : \Omega \to D$ be injective holomorphic such that $f(a) = 0$. Then the map f attain **maximum** of

$$\begin{aligned} \{h \in \mathscr{O}(\Omega, D) \mid h \text{ is injective }, h(a) = 0\} &\to [0, \infty) \\ h &\mapsto |h'(a)| \end{aligned}$$

$\iff$ f is surjective.

uniqueness

For a give $a \in \Omega$, let there are two biholomorphisms

$$f, g : \Omega \to D$$

then

$$f \circ g^{-1} : D \to D$$

is a automorphism of the disk such that

$$f \circ g^{-1}(0) = f(a) = 0$$
$$(f \circ g^{-1})'(0) = f'(g^{-1}(0)) \frac{1}{g'(g^{-1}(0))} < 1$$

- By

  (Swartz inequality)

  $$f \in \mathscr{O}(D, D), f(0) = 0 \implies \begin{cases} |f'(0)| \leq 1 \\ |f(z)| \leq |z| \end{cases}$$

  Moreover, equality in *either* of these inequalities $\iff$ its a rotation $f(z) \equiv cz, |c| = 1$.

  we conclude

  $$f \circ g^{-1}(w) = cw, |c| = 1$$

- This means

  $$f(z) = cg(z)$$

  but

  $$0 < f'(a) = cg'(a), g'(a) > 0 \implies c = 1$$

  thus $f = g$.

[1]

constructing a biholomorphism into the unit disk

- Assume $0 \notin \Omega$. Then there exists a branch of square root $\sqrt{}$ on Ω. Let the image be $\sqrt{\Omega}$.
  - The square root

    $$\sqrt{} : \Omega \to \sqrt{\Omega}$$

    is bijective with inverse $z \mapsto z^2$.
  - If $w \in \sqrt{\Omega}$ then $-w \notin \sqrt{\Omega}$, as otherwise there are $z_1, z_2 \in \Omega$ such that

    $$\sqrt{z_1} = w = -\sqrt{z_2} \implies z_1 = z_2 \text{ or } w = -w$$

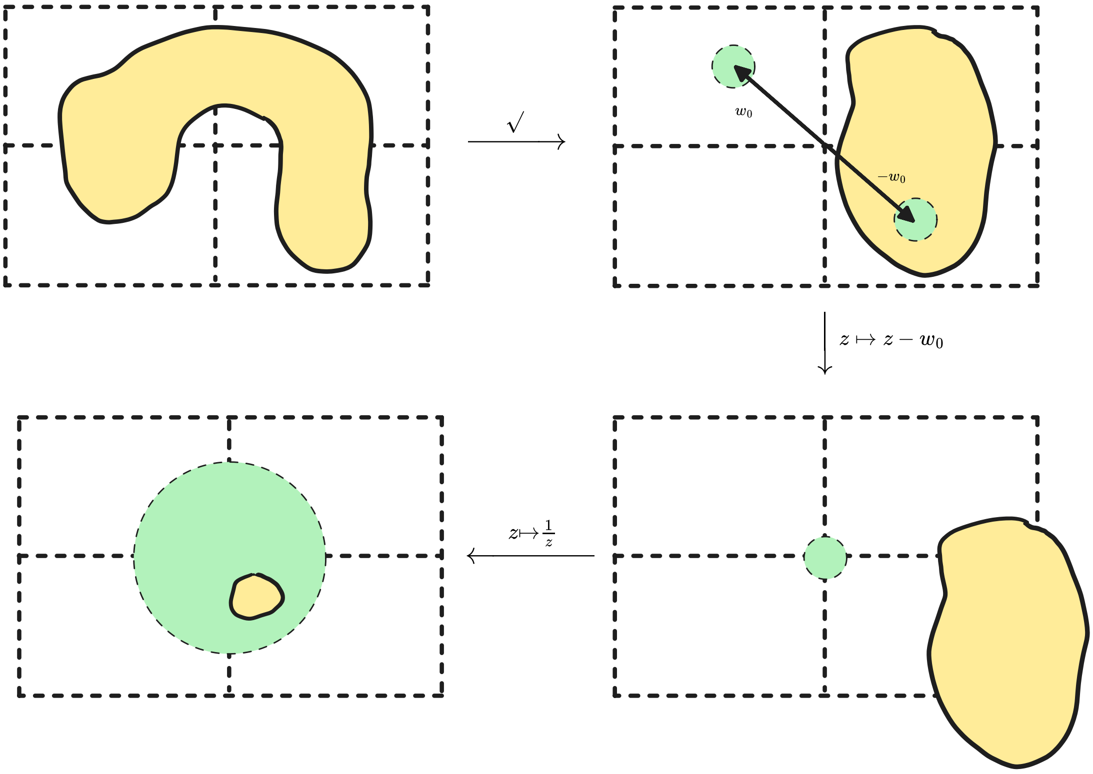
    which implies $0 \in \Omega$ which is a contradiction.

- As $\sqrt{\Omega}$ is open so there is a $B_\delta(-w_0) \subseteq \sqrt{\Omega}$ then

$$\sqrt{\Omega} \cap B_\delta(w_0) = \emptyset$$

- So

$$\begin{aligned} \left|\sqrt{z} - w_0\right| &\geq \delta \\ \frac{\delta}{2} \frac{1}{\left|\sqrt{z} - w_0\right|} &\leq \frac{1}{2} < 1 \end{aligned}$$



-
- Thus

$$\begin{aligned} f : \Omega &\to D \\ z &\mapsto \frac{\delta}{\sqrt{z} - w_0} \end{aligned}$$

is injective into the unit disk D.

subjectivity $\iff$ maximality

- *(surjectivity)* For such an Ω, let

$$f : \Omega \to D$$

be an *injective* holomorphic map such that

$$f'(0) = \sup_{g \in \mathcal{F}} \left|g'(0)\right|$$

Assume $\alpha \in D \setminus f(\Omega)$.

- Let $R_\alpha : D \to D$ be an holomorphic automorphism such that $R_\alpha(\alpha) = 0$, then

  $$0 \notin R_\alpha(f(\Omega))$$

  so $R_\alpha \circ f$ avoids the origin $0 \in D$.
- As Ω is simply connected, we consider a square root

  $$g^2 = R_\alpha \circ f$$

- Then $h := R_{g(0)} \circ g$ is injective holomorphic map that sends $g(0) \mapsto 0$.
- Then

  $$f = R_\alpha^{-1} \circ (\,)^2 \circ R_{g(0)}^{-1} \circ h$$

- By ... we have

  $$\left|(R_\alpha \circ (\,)^2 \circ R_{g(0)}^{-1})'(0)\right| < 1$$

  hence

  $$|f'(0)| < |h'(0)|$$

  which is a contradiction.

- *(maximality)* Let $f : \Omega \to D$ be a bijective holomorphic map with $f(a) = 0$.
  - For any $h : \Omega \hookrightarrow D$ injective holomorphic with $h(a) = 0$ then

    $$g := h \circ f^{-1} : D \to D$$

    with $g(0) = 0$
  - By

    > **(Swartz inequality)**
    >
    > $$f \in \mathscr{O}(D, D), f(0) = 0 \implies \begin{cases} |f'(0)| \leq 1 \\ |f(z)| \leq |z| \end{cases}$$
    >
    > **Moreover, equality in *either* of these inequalities $\iff$ its a rotation $f(z) \equiv cz, |c| = 1$.**

    we have

    $$|g'(0)| \leq 1$$

  - Now

    $$h'(a) = f'(a)g'(0)$$

    so

    $$|h'(a)| \leq |f'(a)|$$

the family of injective holomorphic maps is pre-compact

- *(existance)* We assume $\Omega \subset D$ and $0 \in \Omega$. Consider the family

$$\mathcal{F} := \{f : \Omega \to D \text{ holomorphic, injective} \mid f(0) = 0, f'(0) > 0\}$$

  - In particular, $f'(0) \in \mathbb{R}$. As $\mathsf{Id}_\Omega \in \mathcal{F}$ it is not empty.
  - As

$$|f(z)| < 1 \implies \sup_{z \in D} \sup_{f \in \mathcal{F}} |f(z)| < 1$$

    so $\mathcal{F}$ is uniformly bounded on D.
  - Consider the sequence

$$f_m \in \mathcal{F}$$

    such that

$$f'_m(0) \xrightarrow{m \to \infty} \sup_{f \in \mathcal{F}} |f'(0)|$$

  - By

> **(Montel's theorem) A subset $\mathcal{F} \subseteq \mathscr{O}(U)$ is uniformly bounded**
>
> $$\|\mathcal{F}(K)\|_\infty = \sup_{f \in \mathcal{F}} \|f|_K\|_\infty = \sup_{z \in K} \sup_{f \in \mathcal{F}} |f(z)| < \infty$$
>
> **on every compact subset $K \subseteq U$ $\iff$ pointwise bounded and equicontinuous**
>
> $$\forall a \in U, \|\mathcal{F}(a)\|_\infty < \infty$$
> $$\forall \epsilon > 0 \exists \delta > 0 : \forall z \in \mathcal{F}(B_\delta(x_0)), d(z, f(x_0)) < \epsilon$$
>
> **$\iff$ it is pre-compact (in the compact-open topology) $\iff$ for every sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U to a holomorphic function in $\mathscr{O}(U)$.**

    there is a subsequence f_{m_n} which converges

$$f_{m_n} \xrightarrow{n \to \infty} f_\infty \in \mathscr{O}(\Omega)$$

    uniformly on compact subsets of Ω, such that

$$f'_\infty(0) = \sup_{f \in \mathcal{F}} |f'(0)|$$

  - Then $f_\infty(0) = 0, f'_\infty(0) > 0$, hence the map is not constant.
  - As $f_\infty(\Omega) \subset \bar{D}$ by

> **(Open mapping theorem for open subsets in $\mathbb{C}$) Let f be *non-constant* holomorphic function on a connected open set U then**

$f(U)$ **is open (and connected).**

non-constant holomorphic function locally looks like w^k on a change of coordinates

Let we have a holomorphic function f on some open subset of $\mathbb{C}$ containing z_0 .

Proposition: If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that

$$h^k = f \text{ on } V$$

for a smaller open V containing z_0.

Proposition: If $f : U \to \mathbb{C}$ is a *non-constant* holomorphic function on a open subset $U \subseteq \mathbb{C}$,

- then around any $z_0 \in U$ there is a holomorphic $g(z)$

$$f(z) = f(0) + (z - z_0)^k g(z) \text{ around } z_0$$
$$g(z_0) \neq 0$$

- Moreover there is a smaller neighborhood around z_0 and a holomorphic $h(z)$ such that

$$f(z) = f(0) + ((z - z_0)h(z))^k \text{ around } z_0$$
$$h(z_0) \neq 0$$

  with further smaller neighborhood around z_0 so that $(z - z_0)h(z)$ is a biholomorphism to its (open) image.

- Now, we write

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \text{ on } z \in B_R(z_0)$$

- If f is **not constant**, there is some smallest $k \geq 1$ (**multiplicity of the zero**) such that $a_k \neq 0$, meaning

$$f(z) = a_0 + (z - z_0)^k g(z) \text{ on } z \in B_R(z_0)$$
$$g(z) = \sum_{n \geq 0} a_{n+k}(z - z_0)^n$$
$$g(z_0) \neq 0$$

- Then by

> **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
>
> $$h^k = f \text{ on } V$$
>
> for a smaller open V containing z_0.

we find a smaller open V containing z_0 and a holomorphic h such that

$$f(z) = a_0 + (z - z_0)^k (h(z))^k \text{ on } z \in V$$
$$h(z_0) \neq 0$$

- Thus we have

$$f(z) = a_0 + (\underbrace{(z - z_0)h(z)}_{h_1(z)})^k \text{ on } z \in V$$

with $h(z_0) \neq 0$, so

$$h_1(z) = (z - z_0)h(z)$$
$$h_1'(z) = h(z) + (z - z_0)h'(z)$$
$$h_1'(z_0) = h(z_0) \neq 0$$

- As h_1 has a non-vanishing derivative at z_0, so there must be *another* smaller open W containing z_0 where $h_1(W)$ is open and

$$h_1 : W \to h_1(W)$$

is a homeomorphism.

- We can choose W so that $h_1(W)$ is a disk centered at $h_1(z_0) = 0$.

> **Proposition:**
>
> $$f : W \to f(W) = a_0 + D$$
> $$f(z) = a_0 + (h_1(z))^k$$
>
> is an **open** map.

- Given open set $A \subseteq W$ its image $h_1(A)$ under the homeomorphism h_1 is open set in a disk containing 0.
- Image of $h_1(A)$ under $z \mapsto z^k$ is open and translation $w \mapsto a_0 + w$ takes open sets to open sets.
- Thus $f(A)$ is open.

$f(\Omega) \subseteq D$, thus we have a holomorphic

$$f : \Omega \to D$$

by solving a Dirichlet problem

[2]

- Let $\Omega \subset D$ and $0 \in \Omega$. We solve the Dirichilet problem for

  $$\log |\zeta| \text{ for } \zeta \in \partial\Omega$$

  and obtain a harmonic

  $$u : \Omega \to \mathbb{R}$$

  such that

  $$u(z) \xrightarrow{z \to \zeta \in \partial\Omega} \log |\zeta|$$

  - Since Ω is simply connected there is a harmonic conjugate v to u, making

    $$f := z \exp(-(u + iv)) : \Omega \to \mathbb{C}$$

    holomorphic on Ω such that

    $$|f(z)| \xrightarrow{z \to \zeta \in \partial\Omega} 1$$

  - Hence, $f(\Omega) \subset \bar{D}$ and by open mapping or maximum modulus $f(\Omega) \subset D$. Thus

    $$f : \Omega \to D$$

  - Applying the argument principle to ...

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1

- mapping
    - onto disk Biholomorphic mapping onto disk

And it has 9 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
            - mapping
                - bd derivative Bounding derivative of holomorphic mappings
                - branch points Branch points and ramification index of holomorphic functions
                - contains disks Disks in holomorphic images
                - fibers Fibers of holomorphic mappings
                - locally poly Non-constant holomorphic functions locally look like w^k and are open mappings
                - onto disk Biholomorphic mapping onto disk
                - pole Holomorphic functions on punchered disk
                - singularities are not isolated Holomorphic function whose singularities are not isolated
                - z2 at all rationals A holomorphic map branched over $\mathbb{Q}$

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1. https://www.math.stonybrook.edu/~bishop/classes/math626.F08/rmt.pdf ↩
2. complex analysis - Solution verification —— Proof of the Riemann mapping theorem using Perron's solution of the Dirichlet problem. - Mathematics Stack Exchange ↩