Info

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Holomorphic functions on punchered disk

$$z_0 + RD^* := \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$$

- Because modulus of non-constant holomorphic functions cannot attain its maximum in the interior of its domain, holomorphic function needs to have its maximum at the boundary or at z_0.
- Can it have maximum at z_0? No!

expressing as a Laurent series

Proposition: Let $z_0 + RD^* := \{z : 0 < |z - z_0| < r\}$ and

$$f : z_0 + RD^* \to \mathbb{C}$$

be holomorphic, then it has a Laurent series

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

which converges in $a + RD^*$ (that means each of the series $\sum_{n \geq 0} a_n z^n, \sum_{n \geq 1} a_{-n} z^{-n}$ converges near $z = 0$) given by

$$a_n = \frac{1}{2\pi i} \int_{S^1_s} \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z$$

singularities

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$$a_n = \frac{1}{2\pi i} \int_{S_s^1} \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z$$

Then there are two cases:

- $n < -k \implies c_n$. Let k be the largest such k, then
  - if $k \geq 0$, the function is analytic, *removable singularity*
  - if $k < 0$, f has a *pole*
- $\{n : c_n \neq 0\}$ is not bounded below, *essential singularity*

- Let

$$\{z : |f(z) - A| < \epsilon\} = \emptyset$$

  for some $A \in \mathbb{C}, \epsilon > 0$
  - Then

$$|f(z) - A| \geq \epsilon \text{ on } U^*$$

    which means

$$h(z) := \frac{1}{f(z) - A} \text{ on } U^*$$

    is holomorphic and

$$|h(z)| = \left| \frac{1}{f(z) - A} \right| \leq \frac{1}{\epsilon} \text{ on } U^*$$

    which means a is a removable singularity of $h(z)$
  - Then

$$f(z) = A + \frac{1}{h(z)}$$

    is holomorphic which means f cannot have an essential singularity on U^*
- This means if f has an essential singularity, it's image is dense in $\mathbb{C}$

Types

f is holomorphic on $z_0 + RD^*$	type of (isolated) singularities	the Laurent series $f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$	image of f on $z_0 + \mathbb{D}_R^*$
	removable singularity, holomorphic at z_0	$c_n = 0$ for all $n < 0$	bounded
$(z-z_0)^k f(z)$ has a *removable singularity* for some $k > 0$ but f does not	pole, meromorphic at z_0	eventually $c_n = 0$	$\|f(z)\| \to \infty$ as $\|z\| \to 0$
$(z-z_0)^k f(z)$ does **not** have a removable singularity for any $k \in \mathbb{N}$	essential singularity	c_n does not become 0 eventually	image is **dense** on $\mathbb{C}$ (actually reaches almost all values on $\mathbb{C}$)

The function

$$e^{1/z}$$

does not reach the point 0

⚠ **Warning**

Just because some function has Laurent series, does not mean *residue theorem* works. It works only for isolated singularities, that is, locally must be a function on punchered disk.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - pole Holomorphic functions on punchered disk

And it has 9 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - bd derivative Bounding derivative of holomorphic mappings
        - branch points Branch points and ramification index of holomorphic functions
        - contains disks Disks in holomorphic images
        - fibers Fibers of holomorphic mappings
        - locally poly Non-constant holomorphic functions locally look like w^k and are open mappings
        - onto disk Biholomorphic mapping onto disk
        - pole Holomorphic functions on punchered disk
        - singularities are not isolated Holomorphic function whose singularities are not isolated
        - z2 at all rationals A holomorphic map branched over $\mathbb{Q}$