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A holomorphic map branched over $\mathbb{Q}$

Let $\Omega := \mathbb{C} \setminus \{0\}$ and $B := \left\{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\right\}$ so set of limit points of B is $\{0\}$ which is outside Ω. Now consider an enumeration (q_n) of $\mathbb{Q}$ and the polynomials

$$P_{\frac{1}{n}}(z) := q_n + \frac{1}{2}z^2$$

By

Let Ω be a open subset of $\mathbb{C}$ and $B \subseteq \Omega$ with no limit points in Ω. For each $b \in B$ let $P_b \in \mathbb{C}[z]$ be a polynomial. Then there is a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that the *truncated (first few terms of the) Taylor series* of f at b matches the polynomial P_b, that is

$$\sum_{k \leq \deg P_b} \frac{f^{(k)}(b)(z-b)^k}{k!} = P_b(z-b)$$

for each $b \in B$.

there exists a holomorphic function

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$$

such that

$$f\left(\frac{1}{n}\right) = q_n$$

the Taylor series of f at $\frac{1}{n}$ is $P_{\frac{1}{n}}\left(z - \frac{1}{n}\right)$.

So in particular

- $$f'\left(\frac{1}{n}\right) = 0, f''\left(\frac{1}{n}\right) = 1$$

Thus we observe every point of

$$\mathbb{Q} \subseteq f(\mathbb{C} \setminus \{0\})$$

is a **branch point** of f [1]

[2]

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - z2 at all rationals A holomorphic map branched over $\mathbb{Q}$

And it has 9 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - mapping
        - bd derivative Bounding derivative of holomorphic mappings
        - branch points Branch points and ramification index of holomorphic functions
        - contains disks Disks in holomorphic images
        - fibers Fibers of holomorphic mappings
        - locally poly Non-constant holomorphic functions locally look like w^k and are open mappings
        - onto disk Biholomorphic mapping onto disk
        - pole Holomorphic functions on punchered disk
        - singularities are not isolated Holomorphic function whose singularities are not isolated
        - z2 at all rationals A holomorphic map branched over $\mathbb{Q}$

1. **Definition. Branch points and ramification index of holomorphic functions**

Let $\Omega \subseteq \mathbb{C}$ be a connected open set and

$$f : \Omega \to \mathbb{C}$$

be a non-constant holomorphic function. Then zeros of f', that is, $(f')^{-1}(0)$ are the set of *critical points* of f and in particular this set has no limit points in Ω as f is holomorphic.

The **ramification index** of f is equivalently

- Around a small disk $B(z_0, r)$ of such a zero point $z_0 \in (f')^{-1}(0)$ which does not contain any other zeros, we have the winding number of the curve $f(\gamma)$
- order of the zero z_0 of f'

If the ramification index at z_0 is greater than 1 then z_0 is called a **ramification point** of f, the *critical value* $f(z_0)$ is called an **(algebraic) branch point** of f.

↩

2. https://mathoverflow.net/a/60681 ↩