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Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1

Definition. Holomorphic functions on $\mathbb{C}$

Given an open subset $U \subseteq \mathbb{C}$, a function

$$f : U \to \mathbb{C}$$

is called **holomorphic** if it satisfies any of the equivalent conditions

- $$\lim_{h \to 0, h \in \mathbb{C} \setminus \{0\}} \frac{f(z+h) - f(z)}{h}$$

  exists for all point $z \in U$
- $\iff$ f is *analytic* that is $\forall z_0 \in U, \exists \epsilon > 0$ such that

  $$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

  for all $z \in B(\epsilon, z)$
- $\iff$ f satisfies the Cauchy-Riemann equations

  $$\frac{\partial}{\partial \bar{z}} f = 0$$

- $\iff$ it satisfies Cauchy integral formula, that is, $f \in C^1(U)$, for all $z_0 \in U$, $\exists \epsilon > 0$

  $$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(\epsilon, z_0)} \frac{f(z)}{z - z_0}$$

- $\iff$ the complexified total derivative (whose Jacobian is)
  $\mathfrak{D}^{\mathbb{C}}_{z_0} f : T_{z_0}\mathbb{R}^2 \otimes \mathbb{C} \to T_{f(z_0)}\mathbb{R}^2 \otimes \mathbb{C}$

  $$\begin{bmatrix} \frac{\partial}{\partial z} f & \frac{\partial}{\partial \bar{z}} f \\ \underbrace{\frac{\partial}{\partial z} \bar{f}}_{\overline{\frac{\partial}{\partial \bar{z}} f}} & \underbrace{\frac{\partial}{\partial \bar{z}} \bar{f}}_{\overline{\frac{\partial}{\partial z} f}} \end{bmatrix}$$

is diagonal $\iff$ it is a $\mathbb{C}$-linear map and then we have the complex total derivative (whose Jacobian is)

$$[\frac{\partial}{\partial z}f]$$

object	**some derivative operators**	**equivalent to Cauchy-Riemann operator**	**Cauchy-Riemann equations is when the Cauchy-Riemann operator gives 0**	**integration**	**primitive**	**what does premitive satisfy**
complex function $f = u + iv$	The *total derivative* $\mathfrak{D}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ which splits into $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$	$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\Big($	Holomorphic		a holomorphic g such that $\frac{\partial g}{\partial z} = f$	
a (complex) differential $(1,0)$-form $f(z)\mathrm{d}z$		$\mathrm{d}(f(z)\mathrm{d}z) =$	closed differential form	$\int f(z)\mathrm{d}z = \int (u + iv)$ $= \int (u\mathrm{d}x -$	an holomorphic g such that $\mathrm{d}g = f\mathrm{d}z$	
(real) differential 1-form $\omega = u\mathrm{d}x - v$ and its Hodge dual $\star\omega = v\mathrm{d}x +$		$\mathrm{d}\omega = (u_y +$ $\mathrm{d}\star\omega = (v_y -$	closed, coclosed differential form	$\int \omega + i \int$ $= \int (u\mathrm{d}x -$	a smooth ϕ such that $\mathrm{d}\phi = \omega$	$\mathrm{d} \star \mathrm{d}\phi = 0$

object	some derivative operators	equivalent to Cauchy-Riemann operator	Cauchy-Riemann equations is when the Cauchy-Riemann operator gives 0	integration	primitive	what does premitive satisfy
(real) vector field $\bar{f} = \begin{bmatrix} u \\ -v \end{bmatrix}$		$\text{curl}\bar{f} = -v_x$ $\text{div}\bar{f} = u_x$	*incompressible* (area preserving/ symplectic) , *solenoid* vector field	$\int \bar{f} \cdot (\gamma + i\gamma$?	a smooth ϕ such that $\bar{f} = \text{grad}(\phi)$	$\Delta\phi = 0$
two real functions $u(x, y), v(x,$	Δ Δ $\text{grad}u \cdot \text{grad}$	$\text{grad}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$	gradient of u rotated 90 degrees is the gradient of v - both are harmonic and conjugate to each other			
one real function $u(x, y)$			orthogonal real (harmonic?) functions $\lVert\text{grad}u\rVert_{z_0} =$ at some $z_0 \in \mathbb{C}$			

- stamp.Rf.1Hol.space
- stamp.Rf.1Hol.sheaf
- Derivative of holomorphic functions
- stamp.Rf.1Hol.Hol.integral
- stamp.Rf.1Hol.Hol

on a	antiderivative	local	global
any open set	no	non-constant holomorphic function locally looks like w^k on a change of coordinates	
connected open sets	no		
space.R.2.disk.Hol	yes	**(Open mapping theorem for open subsets in $\mathbb{C}$) Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).** non-constant holomorphic function locally looks like w^k on a change of coordinates Let we have a holomorphic function f on	Because modulus of non-constant holomorphic functions cannot attain its maximum in the interior of its domain, on the disk D_R of radius R, a holomorphic function that is continuous on the closed disk needs to have its

on a	antiderivative	local	global
		some open subset of $\mathbb{C}$ containing z_0. **Proposition:** If $f(z_0) \neq$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a functi	maximum at the boundary $S^1_{(R)}$.

on a	antiderivative	local	global
		on h such that $h^k = f \circ$ for a smaller open V containing z_0. --- **Proposition:** If $f : U \to$ is a *non-constant* hol	

on a	antiderivative	local	global
		omorphic function on a open subset $U \subseteq \mathbb{C}$, • then around any $z_0 \in$ there is a hol	

on a	antiderivative	local	global
		omorphic $g(z)$ $f(z)$ • Moreover there is a smaller neighb	

on a	antiderivative	local	global
		orhood around z_0 and a holomorphic $h(z)$ such that $f(z)$ w	

on a	antiderivative	local	global
		ith further smaller neighborhood around z_0 so tha	

on a	antiderivative	local	global
		t $(z -$ is a biholomorphism to its (open) image. • Now, we write	

on a	antiderivative	local	global
		$f(z) = \sum_{n\geq 0}$ … • If f is **not constant**, there is some smallest $k \geq 1$ (**multiplicity of the zero**) such that $a_k \neq 0$, meaning $f(z) = a_0 +$ … • Then by **Proposition:** If $f(z$ … and an …	

on a	antiderivative	local	global
		y positive integer k, we compose f with the functi	

on a	antiderivative	local	global
		on $z^{1/}$ to obtain a function h such that h^k for a smalle	

on a	antiderivative	local	global
		r o p e n V c o n t a i n i n g z_0 . we find a smaller open V containing z_0 and a holomorphic h such that $f(z) = a_0 +$ • Thus we have $f(z) =$ with $h(z_0)$	

on a	antiderivative	local		global
			, so $h_1(z)$ $h_1'(z)$ $h_1'(z_0)$ • As h_1 has a non-vanishing derivative at z_0, so there must be *another* smaller open W containing z_0 where $h_1(W)$ is open and $h_1 : W$ is a homeomorphism. • We can choos	

on a	antiderivative	local		global
			e W so that $h_1(W)$ is a disk cente red at $h_1(z_0)$. **Proposition:** $f : W \to$ $f(z) =$ is an **open** map. Given open set $A \subseteq W$ its image $h_1(A)$ under the homeomor phism h_1 is open set in a disk containing 0. • Image of $h_1(A)$	

on a	antiderivative	local	global
		under $z \mapsto z^k$ is open and translation $w \mapsto a_0 + u$ takes open sets to open sets. • Thus $f(A)$ is open. ∎	
open simply connected (biholomorphic to a disk)	yes		
upper half plane (biholomorphic to a disk)	yes		
punctured disks stamp.Rf.1Hol.mapping.pole	no	**Proposition:** Let $z_0 + RD^* :=$ and $f : z_0 + RD^*$ be holomorphic, then it has a Laurent series $f(z) = \sum_{n\in\mathbb{Z}} a_n$ which converges in $a + RD^*$	<table><tr><th>f is holomorphic on $z_0 +$</th><th>type of (isolated) singularities</th><th>the Laurent series $f(z)$</th><th>image of f on $z_0 +$</th></tr><tr><td></td><td>removable singularity, holom</td><td>$c_n =$ for all $n <$</td><td>bounded</td></tr></table>

on a	antiderivative	local	global			
		(that means each of the series $\sum_{n\geq 0} a_n z^n, \sum$ converges near $z = 0$) given by $a_n = \frac{1}{2\pi i} \int_{S_s^1}$		orphic at z_0		
			$(z -$ has a *removable singularity* for some $k >$ but f does not	pole, meromorphic at z_0	eventually $c_n =$	$\|f(z$ as $\|z\|$
			$(z -$ does **not** have a removable	essential singularity	c_n does not become 0 eventually	image is **dense** on $\mathbb{C}$ (actually rea

on a	antiderivative	local	global
			singularity for any $k \in$	ches almost all values on $\mathbb{C}$)
annulus of width R	no		
$\mathbb{C}$ stamp.Rf.1Hol.space $\mathbb{C}$	yes		bounded $\implies$ constant

properties of holomorphic functions

- stamp.Rf.1Hol.Hol.power series
- stamp.Rf.1Hol.Hol.Cauchy
- stamp.Rf.1Hol.Hol.integral
- Non-constant holomorphic functions locally look like w^k and are open mappings
- Zeros and singularities of holomorphic functions
- Sequences of holomorphic functions
  - Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
- Sheaf of holomorphic functions on $\mathbb{C}$
- factorizing, constructing functions
  - Reconstructing meromorphic functions from stalks
  - Factorization of holomorphic functions on $\mathbb{C}$
- extension
  - stamp.Rf.1Hol.reflection
- stamp.Rf.1Hol.Mer
- stamp.Rf.1Hol.Mer at infinity
- stamp.Rf.1Hol.mapping.singularities are not isolated
- stamp.Rf.1Hol.mapping.branch points
- Global holomorphic functions
- stamp.Rf.1Hol.mapping.onto disk

- stamp.Rf.1Hol.rotation symmetrizer

- stamp.Rf.1Hol.modular

- Global holomorphic functions
- polynomials
- rational functions

File (11)	title	Riemann surface
space.R.CP 1.Aut	$PSL(2)(\mathbb{C}) \curvearrowright \mathbb{C}P^1 \cong_{\text{Man}} S^2$	-
stamp.Rf.1Hol.mapping.z2 at all rationals	A holomorphic map branched over $\mathbb{Q}$	-
stock.C1 Hol cos of sqrt	$$\cos\sqrt{z}$$	-
stock.C1 Hol log	The complex logarithm	space.R.2.C minus 1.covering univ
stock.C1 Hol sqrt	$\sqrt{z}$	-
stock.C1 Hol to n factorial	$\sum_{n\geq 0} z^{n!}$	-
stock.C1 Hol Weierstrass elliptic	Weierstrass elliptic function $\wp$	-
stock.C1 Hol z over z2 -1	$\frac{z}{z^2-1} = \frac{z}{(z+1)(z-1)}$	$$\mathbb{C}\setminus\{-1,1\}$$
stock.C1 Hol z3 - 3z	$z^3 - 3z$	-
stock.exp	The complex exponential $\exp(z)$	Holomorphic on $\mathbb{C}$
stock.exp of z+1 by z-1	$\exp\left(\frac{z+1}{z-1}\right)$	-

- combinatorics - Why does this random walk generating function admit an analytic continuation? - Mathematics Stack Exchange

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Current note has 22 direct children and 37 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
        - Cauchy Cauchy integral formula for holomorphic functions
        - d Derivative of holomorphic functions
        - integral Integral of holomorphic differential forms
        - power series Convergent power series
      - mapping
        - bd derivative Bounding derivative of holomorphic mappings
        - branch points Branch points and ramification index of holomorphic functions
        - contains disks Disks in holomorphic images
        - fibers Fibers of holomorphic mappings
        - locally poly Non-constant holomorphic functions locally look like w^k and are open mappings
        - onto disk Biholomorphic mapping onto disk
        - pole Holomorphic functions on punchered disk
        - singularities are not isolated Holomorphic function whose singularities are not isolated
        - z2 at all rationals A holomorphic map branched over $\mathbb{Q}$
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
        - pre-cpt Pre-compact subsets of $\mathscr{O}(U)$
        - seq Sequences of holomorphic functions
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$
      - space D closed $\mathscr{O}(\overline{D})$

- space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
- space D L2 $\mathscr{O} \cap L^2(D)$
- space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
- space Lp $\mathscr{O} \cap L^p$
- space S1 $\mathscr{O}(S^1)$
- zeros and singularities Zeros and singularities of holomorphic functions

And it has 36 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - circle packing Circle packing on $\mathbb{R}^2$
    - circle packing to Riemann map Circle packing converges to the Riemann biholomorphism
    - Cn conn open bounded Bounded connected open subsets of $\mathbb{C}^n$
    - Cn conn open circular Connected circular open subsets of $\mathbb{C}^n$
    - cont Continuous functions on $\mathbb{R}^d$
    - cube dyadic Dyadic cubes
    - curves Curves
    - derivative Differentiable functions
    - forms Differential forms on $\mathbb{R}^n$
    - Fourier-Wigner Fourier-Wigner transform
    - harmonic composed conformal Harmonic functions composed with conformal maps
    - Hilbert Hilbert transform
    - hol harmonic disk-circle Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
    - Hol sets Holomorphic subsets of $\mathbb{C}^n$
    - hypersurf 2n reg Regular hypersurfaces in $\mathbb{R}^{2n}$
    - hypersurf or Orientable hypersurfaces in $\mathbb{R}^n$
    - KG

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2 + m^2$$

    - Laplace Laplace operator on $\mathbb{R}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
    - Lmeas bd of open Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
    - met density Metric density of subsets of $\mathbb{R}^n$
    - Mobius n-sphere Mobius endomorphisms

- monotone Monotone functions on $\mathbb{R}$
- periodic int Cauchy Cauchy integral of periodic functions
- poly int Polygons with integer vertices
- R 2 open smooth End Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- R n discrete subg Discrete subgroups of $\mathbb{R}^n$
- R n discrete subg cocpt Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- RC ramified germs Ramified germs of smooth and holomorphic functions
- Rn open Open subsets of $\mathbb{R}^n$
- Rn open Riem Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- smooth quasi-analytic Quasi-analytic smooth functions on $\mathbb{R}$
- star shaped Star-shaped subsets of $\mathbb{R}^n$
- Vec ODEs in $\mathbb{R}^n \leftrightarrow$ Vector fields in $\mathbb{R}^n$
- wave

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2$$