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is written (completely with human hands) by Rupadarshi Ray,
created on April 9, 2024 2:51:45 PM,
and was last modified on June 13, 2026 7:06:57 PM.

Reconstructing meromorphic functions from stalks

(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A \subseteq \Omega$ with no limit point in Ω. For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a.

Ash and Novinger - Complex Variables , p.144

For such a f, the reciprocal

$$\frac{1}{f} : \Omega \setminus A \to \mathbb{C}$$

will be holomorphic with poles of order m_a precisely at every $a \in A$.

reconstructing meromorphic function from its stalks at poles

(Mittag-Leffler's theorem for open sets on $\mathbb{C}$) Let $U \subseteq \mathbb{C}$ be open and $E \subseteq U$ with its limit points in ∂U. For $a \in E$ let p_a be a polynomial with $p_a(0) = 0$. Then there exists a unique meromorphic function up to addition of a holomorphic function

$$f \in \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$

whose poles are precisely the elements of E and for each such pole $a \in E$, the

function

$$f(z) - p_a\left(\frac{1}{z-a}\right)$$

has a *removable singularity* at a, that is the principal part of f at a is $p_a\left(\frac{1}{z-a}\right)$.

If the "principal part", which are elements of the quotient of stalks at $a \in E$

$$p_a\left(\frac{1}{z-a}\right) \in \frac{\mathrm{Mer}_a}{\mathrm{Hol}_a}$$

is given, then there is an unique

$$f \in \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$

which has these principal parts.

! How nice is this give and take? Claim: Fix such a discrete $E \subseteq U$ then

$$\mathrm{Mitt} : \bigoplus_{a \in E} \frac{\mathrm{Mer}_a}{\mathrm{Hol}_a} \to \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$

is a linear map?, whose image is the space of all maps whose poles are exactly the elements of E. The inverse of this map is just mapping a meromorphic f on U to its stalk at $a \in U$. The first one is just polynomials in $\frac{1}{z-a}$

$$\bigoplus_{a \in E} \mathbb{C}\left[\frac{1}{z-a}\right]$$

?

reconstructing holomorphic functions from first few terms of its Taylor series at a discrete set of points

Let Ω be a open subset of $\mathbb{C}$ and $B \subseteq \Omega$ with no limit points in Ω. For each $b \in B$ let $P_b \in \mathbb{C}[z]$ be a polynomial. Then there is a holomorphic function

$$f : \Omega \to \mathbb{C}$$

such that the *truncated (first few terms of the) Taylor series* of f at b matches the polynomial P_b, that is

$$\sum_{k \leq \deg P_b} \frac{f^{(k)}(b)(z-b)^k}{k!} = P_b(z-b)$$

for each $b \in B$.

Apply

> **(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A \subseteq \Omega$ with no limit point in Ω. For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function**
>
> $$f : \Omega \to \mathbb{C}$$
>
> **such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a.**
>
> Ash and Novinger - Complex Variables , p.144

to produce a holomorphic function $g : \Omega \to \mathbb{C}$ such that $f^{-1}(0) = B$ and for each $b \in B$ the multiplicity of f at b is $m_b := \deg(P_b) + 1$.

- Let $m \geq 1$ and

  $$g(z) = \sum_{k \geq m} a_k (z-b)^k$$

  be holomorphic at b and

  $$R(z) = \frac{c_1}{z-b} + \cdots + \frac{c_m}{(z-b)^m}$$

  be a rational function. Then gR has a removable singularity at b so

  $$gR(z) = d_0 + d_1(z-b) + d_2(z-b)^2 + \ldots$$

  where

  $$\begin{aligned} d_0 &= a_0 c_m \\ d_1 &= a_0 c_{m-1} + a_1 c_m \end{aligned}$$

  - For given $c_1, c_2 \ldots, c_m$ we may determine $d_0, d_1, \ldots, d_{m-1}$.
  - Conversely, if g is given as above (order of zero at b is ≥ 1) and $d_0, \ldots d_{m-1}$ are given complex number, because $a_0 \neq 0$ we may solve the equations

    $$\begin{aligned} c_m &= \frac{d_0}{a_0} \\ c_{m-1} &= \frac{d_1}{a_0} - \frac{a_1}{a_0} c_m \end{aligned}$$

    inductively, to obtain, in order, $c_m, c_{m-1}, \ldots, c_1$. Thus we constructed R such that

    $$(gR)(z) = \underbrace{d_0 + d_1(z-b) + \cdots + d_{m-1}(z-b)^{m-1}}_{\text{given}} + \text{higher order terms } \ldots$$

- Hence, we have rational functions R_b for each $b \in B$ such that

  $$g(z) R_b(z) = P_b(z) + \text{higher order terms } \ldots$$

near b.

- Finally, by

> **(Mittag-Leffler's theorem for open sets on $\mathbb{C}$) Let $U \subseteq \mathbb{C}$ be open and $E \subseteq U$ with its limit points in ∂U. For $a \in E$ let p_a be a polynomial with $p_a(0) = 0$. Then there exists a unique meromorphic function up to addition of a holomorphic function**
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> $$f \in \frac{\mathrm{Mer}(U)}{\mathrm{Hol}(U)}$$
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> **whose poles are precisely the elements of E and for each such pole $a \in E$, the function**
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> $$f(z) - p_a\left(\frac{1}{z-a}\right)$$
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> **has a *removable singularity* at a, that is the principal part of f at a is $p_a\left(\frac{1}{z-a}\right)$.**

  we obtain h, meromorphic on Ω, such that for each b

  $$h - R_b$$

  has a *removable singularity* at each b.

- Now around each b,

  $$gh = g\underbrace{(h - R_b)}_{\text{hol}} + \underbrace{gR_b}_{\text{hol}}$$
  $$gh(z) = (z^{\deg P_b + 1} + \dots) + P_b(z) + \text{higher order terms } \dots$$

  Hence, gh extends to a holomorphic function on Ω which has the required properties.

[1]

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - recons mer from stalk Reconstructing meromorphic functions from stalks

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions

- embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
- factorization Factorization of holomorphic functions on $\mathbb{C}$
- global Global holomorphic functions
- Hol Equivalent descriptions of holomorphic functions
- Mer Meromorphic functions on the complex plane
- Mer at infinity Holomorphic functions meromorphic at infinity
- modular Modular forms
- recons mer from stalk Reconstructing meromorphic functions from stalks
- reflection Extending holomorphic functions by reflections
- rotation symmetrizer Rotational symmetrization of holomorphic functions
- sheaf Sheaf of holomorphic functions on $\mathbb{C}$
- space $\mathscr{O}(U)$
- space C $\mathscr{O}(\mathbb{C})$
- space D $\mathscr{O}(D)$
- space D closed $\mathscr{O}(\overline{D})$
- space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
- space D L2 $\mathscr{O} \cap L^2(D)$
- space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
- space Lp $\mathscr{O} \cap L^p$
- space S1 $\mathscr{O}(S^1)$
- zeros and singularities Zeros and singularities of holomorphic functions

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1. Wayback Machine 6.3.3 ↩