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Rotational symmetrization of holomorphic functions

Only sum terms that are multiples of N in $\mathbb{C}[[z]]$

#wiki/genfx

- Let

$$f(z) = \sum_{n \geq 0} a_n z^n$$

  be a formal power series in complex entries, that is $f(z) \in \mathbb{C}[[z]]$.
- Fix a positive integer N. Let ζ_N be a N-th root of unity.
  - Then if $n \in N\mathbb{N}$ then

$$\sum_{k=1}^{N} \zeta_N^{nk} = \sum_{k=1}^{N} (1)^k = N$$

  - If $n \in \mathbb{N} \setminus N\mathbb{N}$ then

$$\sum_{k=0}^{N-1} (\zeta_N^n)^k = 1 + \zeta_N + \zeta_N^2 + \cdots + \zeta^{N-1} = \frac{1 - \cancelto{1}{\zeta^{nN}}}{1 - \zeta^n} = 0$$

- Then

$$\begin{aligned}
f(z) &= a_0 + a_1 z + a_2 z^2 + \dots \\
f(\zeta_N z) &= a_0 + a_1 \zeta_N z + a_2 \zeta_N^2 z^2 + \dots \\
f(\zeta_N^2 z) &= a_0 + a_1 \zeta_N^2 z + a_2 \zeta_N^4 z^2 + \dots \\
f(\zeta_N^3 z) &= a_0 + a_1 \zeta_N^3 z + a_2 \zeta_N^6 z^2 + \dots \\
\dots & \\
f(\zeta_N^{N-1} z) &= a_0 + a_1 \zeta_N^{N-1} z + a_2 \zeta_N^{2N-2} z^2 + \dots
\end{aligned}$$

- which means

$$\sum_{k=1}^{N} f(\zeta_N^k z) = \sum_{n\geq 0} a_n \left(\sum_{k=1}^{N} \zeta_N^{nk}\right) z^n = \sum_{l\geq 0} a_{lN} N z^{lN}$$

$$\frac{1}{N}\sum_{k=1}^{N} f(\zeta_N^k z) = \sum_{l\geq 0} a_{lN} z^{lN}$$

$$= a_0 + a_N z^N + a_{2N} z^{2N} + \dots$$

- Moreover

$$\frac{1}{N}\sum_{k=1}^{N} f(\zeta_N^k \sqrt[N]{z}) = \sum_{l\geq 0} a_{lN} z^l$$

$$= a_0 + a_N z + a_{2N} z^2 + \dots$$

Then if $f : U \to \mathbb{C}$ is holomorphic on a open subset U of $\mathbb{C}$ and

$$f(z) = \sum_{n\geq 0} a_n z^n$$

on a disk, then call

$$\frac{1}{N}\sum_{k=1}^{N} f(\zeta_N^k z) = \sum_{l\geq 0} a_{lN} z^{lN}$$

the **$\mathbb{Z}_N$-symmetrizer of f**.

Proposition: The $\mathbb{Z}_N$-symmetrizer of f is invariant under the $\mathbb{Z}_N$ action on $\mathbb{C}$.

Call the function

$$\frac{1}{N}\sum_{k=1}^{N} f(\zeta_N^k \sqrt[N]{z}) = \sum_{l\geq 0} a_{lN} z^l = a_0 + a_N z + a_{2N} z^2 + \dots$$

the **rooted $\mathbb{Z}_N$-symmetrizer of f** which is also holomorphic $U \to \mathbb{C}$.

Example

The rooted $\mathbb{Z}_2$ rotational symmetrizer of $\cos z$ is $\cos\sqrt{z}$.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - rotation symmetrizer Rotational symmetrization of holomorphic functions

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$
      - space D closed $\mathscr{O}(\overline{D})$
      - space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
      - space D L2 $\mathscr{O} \cap L^2(D)$
      - space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - space Lp $\mathscr{O} \cap L^p$
      - space S1 $\mathscr{O}(S^1)$
      - zeros and singularities Zeros and singularities of holomorphic functions