> **Info**
>
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> is written (completely with human hands) by Rupadarshi Ray,
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$\mathscr{O}(U)$

Let $U \subseteq \mathbb{C}$ be open and

$$\mathscr{O}(U)$$

be the $\mathbb{C}$-algebra of holomorphic functions.

$\mathscr{O}(U) \subseteq \mathcal{C}(U, \mathbb{C})$

From

> **Definition. $\mathcal{C}(U, X)$**
>
> Let (X, d) be a complete metric space, $U \subseteq \mathbb{R}^n$ be a open subset and
>
> $$\mathbf{EndTop}(U, X)$$
>
> be the space of all continuous functions $U \to X$.
>
> - For $K \subseteq U$ compact we have the distance
>
> $$\rho_K(f, g) := \sup \{d(f(z), g(z)) | z \in K\}$$
>
>   for all $f, g \in \mathcal{C}(G, X)$.
> - Then for $U = \bigcup_{n \in \mathbb{N}} K_n$ where K_n is compact and $K_n \subseteq \text{int}(K_{n+1})$ we define the metric
>
> $$\rho(f, g) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\rho_{K_n}(f, g)}{1 + \rho_{K_n}(f, g)}$$
>
>   on $\mathbf{EndTop}(U, X)$.
>
> The (metrizable) topological space $\mathbf{EndTop}(U, X)$ with the topology induced by the metric ρ is denoted $\mathcal{C}(U, X)$.

we have the compact-open topology on

$$\mathscr{O}(U) \subseteq \mathcal{C}(U, \mathbb{C})$$

If $\{f_n\}$ is a sequence in $\mathscr{O}(U)$ and $f \in \mathcal{C}(U, \mathbb{C})$ such that $f_n \xrightarrow{n\to\infty} f$ then $f \in \mathscr{O}(U)$ and all derivatives

$$f_n^{(k)} \xrightarrow{n\to\infty} f^{(k)}$$

for $k \geq 1$. Thus

$$\mathfrak{D} : \mathscr{O}(U) \to \mathscr{O}(U)$$

is a continuous linear map.

connected $U \iff \mathscr{O}(U)$ is an integral domain

An open set $U \subseteq \mathbb{C}$ is connected $\iff \mathscr{O}(U)$ is an integral domain.

- Let

$$f, g : U \to \mathbb{C}$$

be holomorphic on such that

$$fg \equiv 0 \text{ on } U$$

and $f(z_0) \neq 0$ for some $z_0 \in U$. Then $f \neq 0$ on a disk around z_0, and thus $g = 0$ on that disk. Therefore by

(Identity theorem for holomorphic functions on subsets of $\mathbb{C}$) Let

$$f : \Omega(\text{open}) \subseteq \mathbb{C} \to \mathbb{C}$$

be holomorphic. And L be the set of limit points of $f^{-1}(0)$. Then L is both open and closed in Ω.
Therefore if Ω is connected, either $L = \Omega \iff f \equiv 0 \text{ on } \Omega$ or $f^{-1}(0)$ has no limit point in Ω, that is if $f^{-1}(0)$ has a limit point in Ω then $f \equiv 0$.

$$g \equiv 0 \text{ on } \Omega$$

- If U is not connected, define locally constant functions such that are not zero but one of f, g is zero on each component, thus $fg = 0$.

∎

GCDs exist in $\mathscr{O}(U)$

Let

$$\mathcal{F} \subseteq \mathscr{O}(U)$$

be a non-empty subfamily but $\mathcal{F} \neq \{0\}$. We apply

> **(Holomorphic function on open subsets of $\mathbb{C}P^1$ with given zeros and their multiplicities) Let Ω be a proper open subset of $\mathbb{C}P^1$ and $A \subseteq \Omega$ with no limit point in Ω. For each $a \in A$, let $m_a \in \mathbb{Z}_{>0}$ be a positive integer. Then there exists a holomorphic function**
>
> $$f : \Omega \to \mathbb{C}$$
>
> **such that $A = f^{-1}(0)$ and for each $a \in A$ the multiplicity of f at a is m_a.**
>
> Ash and Novinger - Complex Variables , p.144

to obtain $g \in \mathscr{O}(U)$ such that

$$g^{-1}(0) = B := \bigcap \{f^{-1}(0) \mid f \in \mathcal{F}\}$$

and for each $b \in B$ the multiplicity

$$m(g, b) = \min\{m(f, b) \mid f \in \mathcal{F}\}$$

Then $f \in \mathcal{F}$ implies

$$\frac{g}{f}$$

has a removable singularity at every zero of f, thus

$$g \mid f$$

Furthermore, if $h \in \mathscr{O}(U)$ and $h \mid f$ for each $f \in \mathcal{F}$ then $h^{-1}(0) \subseteq B$ and for each $b \in B$

$$m(h, b) \leq \min\{m(f, b) \mid f \in \mathcal{F}\} = m(g, b)$$

Thus

$$\frac{h}{g}$$

extends holomorphically to U and thus $h \mid g$. Thus g is the greatest common divisor of $\mathcal{F}$.

finitely generated ideals are principal

never a PID, never Noetherian

[1]

functor produces an equivalence of categories

Definition.

$$\mathscr{O} : \mathsf{Open}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Alg}_{\mathbb{C}}$$

Proposition: Let U_1, U_2 be open subsets of $\mathbb{C}$.

- Let $f : U_1 \to U_2$ be a holomorphic map. Then the pullback

$$f^* : \mathscr{O}(U_2) \to \mathscr{O}(U_1)$$

  is a unital $\mathbb{C}$-algebra homomorphism. The map f is a biholomorphism $\iff$ f^* is an isomorphism.
- Let $\Phi : \mathscr{O}(U_2) \to \mathscr{O}(U_1)$ be a unital $\mathbb{C}$-algebra homomorphism. Then for $f := \Phi(\mathsf{Id}_{U_2})$ we have

$$f : U_1 \to U_2$$

  and $\Phi = f^*$.
- Therefore, $\mathscr{O}(U_1)$ and $\mathscr{O}(U_2)$ are isomorphic $\iff$ U_1, U_2 are biholomorphic.

[2]

For $z \in \mathbb{C} \setminus U_2$ we have $\mathsf{Id}_{U_2} - z \in \mathscr{O}(U_2)^{\times}$. Then $\Phi(\mathsf{Id}_{U_2} - z) \in \mathscr{O}(U_1)^{\times}$.

- Therefore for $f(x) := \Phi(\mathsf{Id}_{U_2})(x)$ we have

$$f - z \in \mathscr{O}(U_1)^{\times}$$

  implying $z \notin f(D)$.
- Thus

$$f : U_1 \to U_2$$

- Let $g \in \mathscr{O}(U_2)$. Fix $z \in U_1$. Then

$$g - g(f(z))$$

  has a zero at $f(z)$. There exists $h \in \mathscr{O}(U_2)$ such that

$$g - g(f(z)) = (\mathsf{Id}_{U_2} - f(z))h$$

- Thus

$$\Phi(g - g(f(z))) = (f - f(z))\Phi(h)$$

vanishes at z. Hence,

$$\Phi(g)(z) = g(f(z)) = (f^*g)(z)$$

Definition.

$$\mathscr{O} : \mathsf{Open}(\mathbb{C})^{\mathsf{op}} \to \mathsf{Alg}_{\mathbb{C}}$$

is an equivalence of categories onto some full subcategory of $\mathsf{Alg}_{\mathbb{C}}$.

$\mathrm{maxSpec}\mathscr{O}(U) \cong_{\mathsf{Set}} U$?

continuous dual

$$(\mathscr{O}(U), \mathfrak{T}_{\mathrm{cpt}})^* \cong ? \mathscr{O}_0(\mathbb{C}\boldsymbol{P}^1 \setminus U)$$

Current note has 2 direct children and 2 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - space $\mathscr{O}(U)$
        - pre-cpt Pre-compact subsets of $\mathscr{O}(U)$
        - seq Sequences of holomorphic functions

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks

- reflection Extending holomorphic functions by reflections
- rotation symmetrizer Rotational symmetrization of holomorphic functions
- sheaf Sheaf of holomorphic functions on $\mathbb{C}$
- space $\mathscr{O}(U)$
- space C $\mathscr{O}(\mathbb{C})$
- space D $\mathscr{O}(D)$
- space D closed $\mathscr{O}(\overline{D})$
- space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
- space D L2 $\mathscr{O} \cap L^2(D)$
- space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
- space Lp $\mathscr{O} \cap L^p$
- space S1 $\mathscr{O}(S^1)$
- zeros and singularities Zeros and singularities of holomorphic functions

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1. Wayback Machine ↩
2. abstract algebra - Are the rings of holomorphic functions over a domain $D \subset \mathbb{C}$ isomorphic for different D? - Mathematics Stack Exchange ↩