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Pre-compact subsets of $\mathscr{O}(U)$

- Suppose a **pre-compact** subset $\mathcal{F} \subseteq \mathcal{C}(U)$ is **not** uniformly bounded

$$\|\mathcal{F}(K)\|_\infty = \sup_{f\in\mathcal{F}} \|f|_K\|_\infty = \sup_{z\in K}\sup_{f\in\mathcal{F}} |f(z)| < \infty$$

  on every compact subset $K \subseteq U$, so there is a compact $K \subseteq U$ such that

$$\|\mathcal{F}(K)\|_\infty = \infty$$

  - Then there is a sequence $\{f_n\}$ such that

$$\|f_n|_K\| \geq n$$

  - Since $\mathcal{F}$ is pre-compact, we have a convergent subsequence $f_{n_k} \to f_\infty \in \mathcal{C}(U)$, as in the sequence converges uniformly on compact subsets:

$$\|(f_{n_k} - f)|_K\|_\infty \xrightarrow{k\to\infty} 0$$

  - Now

$$\|f|_K\|_\infty + \underbrace{\|(f_{n_k} - f)|_K\|_\infty}_{\xrightarrow{k\to\infty} 0} \geq \underbrace{\|f_{n_k}|_K\|_\infty}_{\xrightarrow{k\to\infty}\infty}$$

    which is a contradiction.
  - Thus, pre-compact subsets are uniformly bounded $\|\mathcal{F}(K)\|_\infty$ on every compact subsets $K \subseteq U$.

uniformly bounded on any $K \subseteq \mathbb{C}\setminus\{0\}$	$\{z \mapsto nz\}$	
not uniformly bounded on any K		

(Montel's theorem) A subset $\mathcal{F} \subseteq \mathscr{O}(U)$ is uniformly bounded

$$\|\mathcal{F}(K)\|_\infty = \sup_{f\in\mathcal{F}} \|f|_K\|_\infty = \sup_{z\in K}\sup_{f\in\mathcal{F}} |f(z)| < \infty$$

on every compact subset $K \subseteq U \iff$ pointwise bounded and equicontinuous

$$\forall a \in U, \|\mathcal{F}(a)\|_\infty < \infty$$
$$\forall \epsilon > 0 \exists \delta > 0 : \forall z \in \mathcal{F}(B_\delta(x_0)), d(z, f(x_0)) < \epsilon$$

$\iff$ it is pre-compact (in the compact-open topology) $\iff$ for every sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U to a holomorphic function in $\mathscr{O}(U)$.

Let $\|\mathcal{F}(K)\|_\infty < \infty$ for every compact $K \subseteq U$.

- Then $\|\mathcal{F}(a)\|_\infty < \infty$ for every $a \in U$. So it is **pointwise bounded**.
- Now, for $a \in U$, we have an $r > 0$ such that

$$\left\|\mathcal{F}(\overline{B_r(a)})\right\|_\infty < \infty$$

Then by

(Cauchy integral formula) Let f be holomorphic in a disk around $z \in \mathbb{C}$ then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{S^1_r} \frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w$$

we have

$$|f(a) - f(z)| \leq \frac{1}{2\pi}\left|\int_{rS^1} \frac{(a-z)f(w)}{(w-a)(w-z)}\mathrm{d}w\right| \leq \frac{4\left\|\mathcal{F}(\overline{B_r(a)})\right\|_\infty}{r}|a-z|$$

- So, for $\epsilon > 0$ letting

$$\delta := \frac{1}{2}\min\left\{\frac{r}{2}, \frac{r\epsilon}{4\left\|\mathcal{F}(\overline{B_r(a)})\right\|_\infty}\right\}$$

we have

$$|a-z| < \delta \implies \forall f \in \mathcal{F}, |f(a) - f(z)| < \epsilon$$

- Hence, $\mathcal{F}$ is **equicontinuous**.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
            - space $\mathscr{O}(U)$
                - pre-cpt Pre-compact subsets of $\mathscr{O}(U)$

And it has 2 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
            - space $\mathscr{O}(U)$
                - pre-cpt Pre-compact subsets of $\mathscr{O}(U)$
                - seq Sequences of holomorphic functions