> **Info**
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> is written (completely with human hands) by Rupadarshi Ray,
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> and was last modified on June 12, 2026 8:03:41 PM.

$\mathscr{O}(\mathbb{C})$

bounded entire functions are constant

- An entire function $f : \mathbb{C} \to \mathbb{C}$ is continuous on every closed disk $\overline{D_R}$ so it **must** attain its maximum on the boundary S^1_R. So for each $z_R \in S^1_R$,

$$z \in \overline{D_R} \implies |f(z)| \leq |f(z_R)|$$

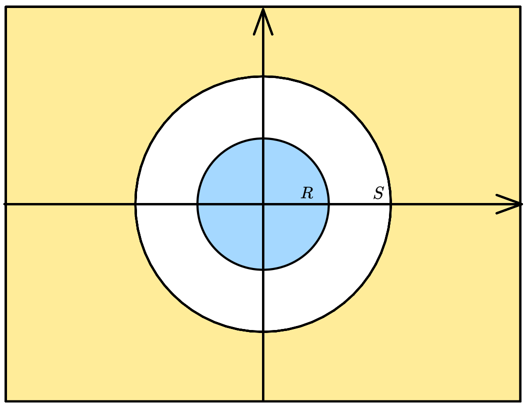
  where the equality only holds if f is constant.

- Suppose $f : \mathbb{C} \to \mathbb{C}$ is **entire** and

$$|f(z)| \xrightarrow{|z| \to \infty} 0 \iff \forall \epsilon \exists \delta : |z| > \delta \implies |f(z)| < \epsilon$$

  then fix $R > 0$ and $z_R \in S^1_R$ and pick $S > R$ such that

$$|z| \geq S \implies |f(z)| \leq \frac{|f(z_R)|}{2}$$



  If $|f(z_R)| > 0$ then inside $D_S \ni z_R$

$$|f(z_R)| \geq \frac{|f(z_R)|}{2}$$

whereas on boundary of $\overline{D_S}$,

$$|f(z)| \leq \frac{|f(z_R)|}{2} \leq |f(z_R)|$$

which is only possible if f is a constant and identically 0.

- If f is **entire** then

$$\frac{f(z) - f(0)}{z}$$

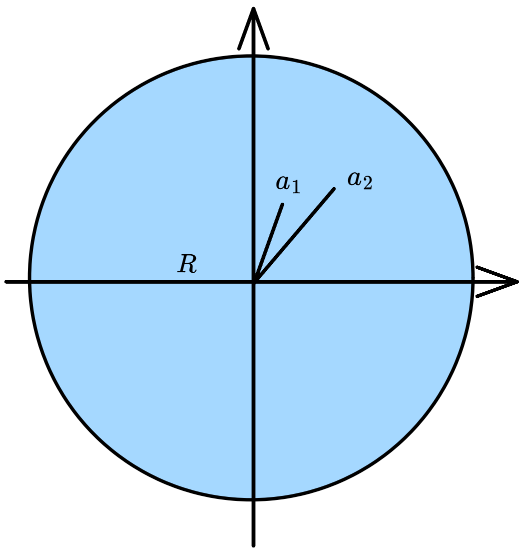
is also entire. If $|f| \leq M$ then

$$\left|\frac{f(z) - f(0)}{z}\right| \leq \frac{2M}{|z|} \implies \left|\frac{f(z) - f(0)}{z}\right| \xrightarrow{|z| \to \infty} 0$$

which by maximum modulus means $f(z) - f(0) = 0$ identically. Thus f is a constant if its bounded and entire.

- Let f is **entire and bounded** by M. For any $a_1, a_2 \in \mathbb{C}$ choose any $R > 0$ large enough so that

$$|z| = R \implies |z - a_i| \leq \frac{R}{2}$$



$$
\begin{aligned}
f(a_i) &= \frac{1}{2\pi i}\int_{S^1_R}\frac{f(z)\mathrm{d}z}{z-a_i}\\
f(a_1)-f(a_2) &= \frac{a_1-a_2}{2\pi i}\int_{S^1_R}\frac{f(z)\mathrm{d}z}{(z-a_1)(z-a_2)}\\
|f(a_1)-f(a_2)| &\le \frac{|a_1-a_2|}{2\pi}\int_{S^1_R}\underbrace{\left|\frac{f(z)\mathrm{d}z}{(z-a_1)(z-a_2)}\right|}_{\le\frac{M}{R^2}}\\
&\le |a_1-a_2|\frac{M}{R}\\
&\xrightarrow{R\to\infty} 0
\end{aligned}
$$

Then $f(a_1)=f(a_2)$ is true for arbitrary $a_1, a_2\in\mathbb{C}$, hence f must be constant.

- Let

$$f=\sum_{n\ge 0}a_n z^n$$

  be **entire**.
  - Consider

$$a_k=\frac{f^{(k)}(0)}{k!}=\frac{1}{2\pi i}\int_{S^1_R}\frac{f(z)}{z^{k+1}}\mathrm{d}z$$

    for any $R>0$.
  - If f is **bounded** by M then for $k>0$

$$|a_k|\le\frac{1}{2\pi}\frac{M}{R^k}\xrightarrow{\to\infty}0$$

  - Thus

$$f(z)=a_0$$

    a constant.

order at infinity

- By
  - Thus for any function f that is meromorphic at z_0, there is a holomorphic $h: z_0+D_r\to\mathbb{C}$ and positive integer k called the **order of the pole at z_0** such that

$$f(z) = \frac{h(z)}{(z-z_0)^k} \text{ on } z_0 + D_r^*$$

Moreover $h(z_0) \neq 0$.

we have for all $w \in D_r^*$

$$f\left(\frac{1}{w}\right) = \frac{h(w)}{w^k} = f(z) = z^n h\left(\frac{1}{z}\right)$$

where h is holomorphic on some D_r, $h(0) \neq 0$ and k is a positive integer called the **order of the pole of f at infinity**.

entire functions with a particular maximum modulus

$$\mathscr{O}(\mathbb{C}) \to \mathcal{C}^\infty(0,\infty)$$
$$f \mapsto |f|^2$$

[1]

entire functions that miss two values is constant

Let

$$f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$$

then because

There is a holomorphic covering

$$D \to \mathbb{C} \setminus \{0,1\}$$

To construct the map, we start with a triangle Δ_0 inside the unit circle consisting of circular arcs that intersect the unit circle at right angles. In other words, a geodesic triangle in the Poincaré disk with vertices "at infinity" (which means that the interior angles are 0). Now reflect Δ_0 across each of its sides. The result are three more triangles with circular arcs intersecting the unit circle at right angles. Figure 4.5 shows how the unit disk is partitioned by triangles as a result of iterating these reflections indefinitely. To obtain the sought after covering map, we start from the Riemann mapping theorem from Chapter 2 which gives us a conformal isomorphism

$f : \Delta_0 \to \mathbb{H}$, the upper half-plane. We may also achieve that the three vertices of Δ_0 get mapped onto $0, 1, \infty$, respectively. Moreover, the map extends as a homeomorphism to the boundary; see Theorem 2.30. Thus, the three circular arcs of Δ_0 get mapped to the intervals $[-\infty, 0], [0, 1], [1, \infty]$, respectively. By the Schwarz reflection principle (see Problem 2.6) the map f extends analytically to the region obtained by reflecting Δ_0 across each of its sides as just explained above. In this way we obtain a conformal map onto $\mathbb{C} \setminus \{0, 1\}$ which is defined on the entire disk. By construction, it is a local isomorphism and a covering map.

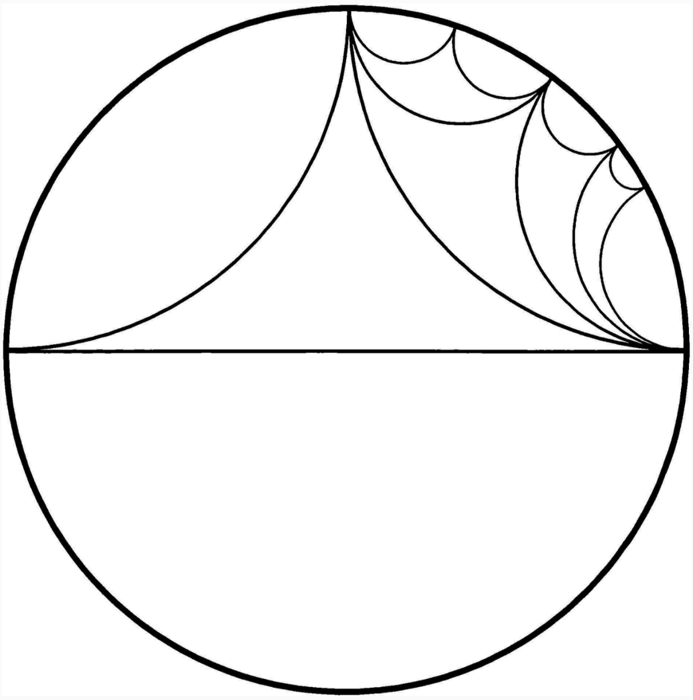


Figure 4.5. Successive reflection of triangles across their sides

∎

we may lift

$$f : \mathbb{C} \to D$$

which is constant because bounded entire functions are constant.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - space C $\mathscr{O}(\mathbb{C})$

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$
      - space D closed $\mathscr{O}(\overline{D})$
      - space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
      - space D L2 $\mathscr{O} \cap L^2(D)$
      - space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
      - space Lp $\mathscr{O} \cap L^p$
      - space S1 $\mathscr{O}(S^1)$
      - zeros and singularities Zeros and singularities of holomorphic functions

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1. citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=6fbe85fa1697efc78b507a1d30a98b5fec32748c ↩