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Zeros and singularities of holomorphic functions

singularities are *not* isolated

under small perturbations

> the number of zeroes inside a simple closed contour is invariant under small perturbations

counting zeros and singularities

- Let holomorphic $f: U(\text{open}) \to \mathbb{C}$ has zeroes and poles in $A \subseteq U$ such that A has no limit points in U. Then $\frac{f'}{f}: U \setminus A \to \mathbb{C}$ is a holomorphic function.
  - Suppose γ is a null homologous path in U then

    $$\frac{1}{2\pi i}\int_\gamma \frac{f}{f'}\mathrm{d}z = \sum_{a \in A} W(\gamma, a)\mathrm{Res}_a\left(\frac{f'}{f}\right)$$

    where the right side is a finite sum as $W(\gamma, a) = 0$ for all but finitely many $a \in A$.
  - Near each $a \in A$, $f(z) = (z-a)^k h(z)$ where $h(a) \neq 0$. It follows then

    $$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \underbrace{\frac{h(z)}{h(z)}}_{\text{holomorphic near } a}$$

    - It follows that $k = \mathrm{Res}_a\left(\frac{f'}{f}\right)$.

Definition. **Order of a function**

Let $\mathrm{ord}_a(f)$ to be the integer k such that $\frac{f(z)}{(z-a)^k}$ extends to a non-vanishing holomorphic function around a. This means k is the order of the zero of f at a, or $-k$ is the order of the pole at a or $f(a)$ is defined and non-zero.

- Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\mathrm{d}z = \sum_{a\in A} W(\gamma, a)\mathrm{ord}_a(f)$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - zeros and singularities Zeros and singularities of holomorphic functions

And it has 22 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
      - approx rat Approximation of holomorphic functions on $\mathbb{C}$ by rational functions
      - embedding into C3 Embedding unit disk D into $\mathbb{C}^3$
      - factorization Factorization of holomorphic functions on $\mathbb{C}$
      - global Global holomorphic functions
      - Hol Equivalent descriptions of holomorphic functions
      - Mer Meromorphic functions on the complex plane
      - Mer at infinity Holomorphic functions meromorphic at infinity
      - modular Modular forms
      - recons mer from stalk Reconstructing meromorphic functions from stalks
      - reflection Extending holomorphic functions by reflections
      - rotation symmetrizer Rotational symmetrization of holomorphic functions
      - sheaf Sheaf of holomorphic functions on $\mathbb{C}$
      - space $\mathscr{O}(U)$
      - space C $\mathscr{O}(\mathbb{C})$
      - space D $\mathscr{O}(D)$

- space D closed $\mathscr{O}(\overline{D})$
- space D cnt bd $\mathscr{O}(D) \cap \mathcal{C}(\overline{D})$
- space D L2 $\mathscr{O} \cap L^2(D)$
- space H $\mathscr{O}^p(H^2_{\mathrm{U}})$
- space Lp $\mathscr{O} \cap L^p$
- space S1 $\mathscr{O}(S^1)$
- zeros and singularities Zeros and singularities of holomorphic functions