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Fourier transform on S^1, Fourier series on $[0,1]$

Definition. **Fourier transform on S^1, Fourier series on $[0,1]$**

Let $f \in L^1(S^1)$. Then for $n \in \mathbb{Z}$ the **n-th Fourier coefficient** of f is

$$\begin{aligned}\hat{f}(n) &:= \frac{1}{2\pi i}\int_{S^1}\frac{f(z)}{z^{n+1}}\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{[-\pi,\pi]} f(e^{i\theta})e^{-in\theta}\frac{\mathrm{d}(e^{i\theta})}{e^{i\theta}}\\ &= \frac{1}{2\pi}\int_{[-\pi,\pi]} f(e^{i\theta})e^{-in\theta}\mathrm{d}\theta\end{aligned}$$

and the $\mathbb{Z}$-sequence f is the **Fourier transform** of f.
Let $f \in L^1[0,1]$. Then for $n \in \mathbb{Z}$ the **n-th Fourier coefficient** of f is

$$\hat{f}(n) := \int_{x\in[0,1]} f(x)e^{-2\pi inx}$$

and the $\mathbb{Z}$-sequence $\hat{f}$ is the **Fourier series** of f.

- For $f \in L^1[0,1]$, the Fourier coefficients of f

  $$\hat{f}(n) := \int_{x\in[0,1]} f(x)e^{-2\pi inx}$$

  is bounded

  $$\begin{aligned}\left|\hat{f}(n)\right| &\leq \left|\int_{x\in[0,1]} f(x)e^{-2\pi inx}\right|\\ &\leq \int_{x\in[0,1]} |f(x)|\\ &= \|f\|_1\end{aligned}$$

  by its L^1-norm.

Actually, the Fourier coeffients converge to 0 at infinity:

(Riemann-Lebesgue lemma for Fourier transform on S^1) Let $f \in L^1(S^1)$ then the Fourier coefficients $\hat{f}(k)$ converge to 0 in both $\mathbb{Z}$-directions

$$\hat{f}(k), \hat{f}(-k) \xrightarrow{k\to\infty} 0$$

That is

$$\widehat{\ } : L^1[0,1] \to \mathcal{C}_0(\mathbb{Z}, \mathbb{C})$$

This does not hold for finite measures.

Definition. **Fourier transform of signed measures on S^1**

Let $\mu \in \mathrm{Meas}(\mathfrak{B}_{[0,1]}, \mathbb{R})$ be a finite $\mathbb{R}$-measure on $[0,1]$. Then

$$\hat{\mu}(n) := \int_{[0,1]} e^{-2\pi i n\theta} \mathrm{d}\mu$$

The Fourier summation

Definition. **N-th partial Fourier summation of f is the trignometric polynomial**

$$(S_N f)(x) := \sum_{-N\le n\le N} \hat{f}(n) e^{2\pi i n x}$$

- Let $f \in \mathcal{C}^{\mathsf{A}}[0,1]$ so that its differentiable almost everyehere and $f' \in L^1[0,1]$.
  - Then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned} \widehat{f}(n) &= \int_{x\in[0,1]} f(x) e^{-2\pi i n x} \\ &= \int_{x\in[0,1]} f(x) \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{-2\pi i n}\frac{\mathrm{d}}{\mathrm{d}x} e^{-2\pi i n x}\right) \\ &= f(0) - f(1) - \frac{1}{-2\pi i n}\int_{x\in[0,1]} f'(x) e^{-2\pi i n x} \end{aligned}$$

  - If $f(0) = f(1)$, that is, $f \in \mathcal{C}^{\mathsf{A}}(S^1)$, then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\widehat{f'}(n) = (2\pi i) n \hat{f}(n)$$

- If $f \in \mathcal{C}^{\mathsf{A}+1}[0,1]$ implying $f^{(2)} \in L^1[0,1]$ and

$$f(0) = f(1), f'(0) = f'(1)$$

then

$$\widehat{f''}(n) = (2\pi i)n\hat{f'}(n) = (2\pi i)^2 n^2 \hat{f}(n)$$

- **Inductively,**

$$\widehat{f^{(k)}}(n) = (2\pi i n)^k \hat{f}(n)$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Fourier
            - S1 Fourier transform on S^1, Fourier series on $[0,1]$

And it has 10 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Fourier
            - L2 bdint Fourier transform on $L^2(-A,A) \leq L^2(\mathbb{R})$
            - L2 Rpos to Hardy2 upper 1 $\widehat{\ } : L^2(0,\infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathrm{U}})$
            - Rn Fourier transform on $\mathbb{R}^n$
            - S1 Fourier transform on S^1, Fourier series on $[0,1]$
            - S1 abs Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$
            - S1 dist Fourier transform of distributions on S^1
            - S1 L1toC0 $\widehat{\ } : L^1[0,1] \to \mathcal{C}_0(\mathbb{Z},\mathbb{C})$
            - S1 L2tol2 $\widehat{\ } : L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z},\mathbb{C})$
            - subsets Fourier transform of measurable subsets
            - unsharp Unsharpness principles