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$\widehat{\quad} : L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$

(Parseval's and Riesz-Fischer theorems) Let $f \in L^2[0,1]$. Then the partial Fourier summation converges to f almost everywhere and in L^2

$$S_N f \xrightarrow{N\to\infty} f \qquad \text{ae and } L^2$$

and the Fourier transform is a $\mathbb{C}$-linear bijection

$$\widehat{\quad} : L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$$

which satisfies $\|f\|_2 = \left\|\hat{f}\right\|_2$ (is an isometry).

trignometric series for an l^2 sequence

- Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$ and consider

$$T_N(x) :\equiv \sum_{-N\leq n\leq N} c_n e^{2\pi i n x}$$

  - The series satisfies

$$\|T_N - T_M\|_2^2 = \sum_{M<m\leq N} |c_m|^2$$

    and thus $\{T_N\}$ is a Cauchy sequence in $L^2[0,1]$.
  - By completeness of $L^2[0,1]$, there is a limit

$$T := \lim_{N\to\infty}^{L^2} T_N$$

    that means

$$\|T - T_N\|_2 \xrightarrow{N\to\infty} 0$$

- Fix $n \in \mathbb{Z}$. For $N > |n|$ we have

$$\begin{aligned} \hat{T}(n) - \hat{T}_N(n) &= \int_{[0,1]} (T - T_N) e_n \\ \implies \left| \hat{T}(n) - c_n \right| &\leq \|T - T_N\|_1 \\ &\leq \|T - T_N\|_2 \\ &\xrightarrow{N \to \infty} 0 \end{aligned}$$

- Therefore,

$$\hat{T}(n) \equiv c_n$$

Hence,

Proposition: Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$. Then

$$\left(\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} \right) \in L^2[0,1]$$

and it's Fourier transform is c itself

$$\widehat{\left(\sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x} \right)}(n) \equiv c_n$$

that is

$$\sum_{n \in \mathbb{Z}} (\ )_n e^{2\pi i n x} : l^2(\mathbb{Z}, \mathbb{C}) \to L^2[0,1]$$

when composed with the Fourier transform is $\mathsf{Id}_{l^2(\mathbb{Z}, \mathbb{C})}$.

Fourier transform of $L^2[0,1]$

- Let $f \in L^2[0,1]$ and consider a trignometric polynomial

$$\sum_{-N \leq n \leq N} b_n e^{-2\pi i n \theta}$$

- Then

$$
\begin{aligned}
&\left\| f - \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\|_2^2 \\
=&\|f\|_2^2 - \left\langle f, \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\rangle \\
&- \left\langle \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta}, f \right\rangle + \left\| \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\|_2^2 \\
=&\|f\|_2^2 - \sum_{-N\le n\le N} \hat{f}(n)\overline{b_n} - \sum_{-N\le n\le N} \overline{\hat{f}(n)} b_n + \sum_{-N\le n\le N} |b_n|^2 \\
=& \sum_{-N\le n\le N} \left| b_n - \hat{f}(n) \right|^2 + \|f\|_2^2 - \sum_{-N\le n\le N} \left| \hat{f}(n) \right|^2
\end{aligned}
$$

- Then the trignometric polynomial for which the *mean square error* $\left\| f - \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\|_2^2$ is mimum must be when

$$
b_n \equiv \hat{f}(n)
$$

And then

$$
\left\| f - \sum_{-N\le n\le N} \hat{f}(n) e^{-2\pi i n\theta} \right\|_2^2 = \|f\|_2^2 - \sum_{-N\le n\le N} \left| \hat{f}(n) \right|^2
$$

- In particular, because $\|\ \|_2 \ge 0$ we have

$$
\sum_{-N\le n\le N} \left| \hat{f}(n) \right|^2 \le \|f\|_2^2
$$

for every $N \ge 0$.

- This implies the sum

$$
\left\| \hat{f} \right\|_2^2 = \sum_{n\in\mathbb{Z}} \left| \hat{f}(n) \right|^2
$$

converges and **(Bessel's inequality)**

$$
\left\| \hat{f} \right\|_2^2 \le \|f\|_2^2
$$

- As

$$
\hat{f} \in l^2(\mathbb{Z}, \mathbb{C})
$$

for $c_n \equiv \hat{f}(n)$, from

> **Proposition:** Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$. Then
>
> $$\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}\right) \in L^2[0,1]$$
>
> and it's Fourier transform is c itself
>
> $$\widehat{\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}\right)}(n) \equiv c_n$$
>
> that is
>
> $$\sum_{n\in\mathbb{Z}} (\ )_n e^{2\pi inx} : l^2(\mathbb{Z}, \mathbb{C}) \to L^2[0,1]$$
>
> when composed with the Fourier transform is $\mathsf{Id}_{l^2(\mathbb{Z},\mathbb{C})}$.

we conclude

$$\widehat{\left(\sum_{m\in\mathbb{Z}} \hat{f}(m) e_m\right)}(n) \equiv \hat{f}(n)$$

- Therefore,

$$\left(\sum_{m\in\mathbb{Z}} \hat{f}(m) e_m\right) \equiv f \qquad \text{ae on } [0,1]$$

and thus

$$S_N f = \sum_{-N\le m\le N} \hat{f}(m) e_m \xrightarrow{N\to\infty} f \qquad \text{ae and } L^2$$

- In particular, taking $\| \ \|_2$ of the converging sequence implies

$$\sum_{-N\le n\le N} \left|\hat{f}(n)\right| \xrightarrow{N\to\infty} \left\|\hat{f}\right\|_2^2 = \|f\|_2^2$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Fourier
      - S1 L2tol2 $\widehat{\ } : L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$

And it has 10 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Fourier
      - L2 bdint Fourier transform on $L^2(-A, A) \leq L^2(\mathbb{R})$
      - L2 Rpos to Hardy2 upper 1 $\frown : L^2(0, \infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathrm{U}})$
      - Rn Fourier transform on $\mathbb{R}^n$
      - S1 Fourier transform on S^1, Fourier series on $[0, 1]$
      - S1 abs Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$
      - S1 dist Fourier transform of distributions on S^1
      - S1 L1toC0 $\frown : L^1[0, 1] \to \mathcal{C}_0(\mathbb{Z}, \mathbb{C})$
      - S1 L2tol2 $\frown : L^2[0, 1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$
      - subsets Fourier transform of measurable subsets
      - unsharp Unsharpness principles