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Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$

The Fourier summation/inversion

$$\begin{aligned} l^1(\mathbb{Z}) &\to \mathcal{C}(S^1) \\ (x_k) &\mapsto \sum_{n\in\mathbb{Z}} x_n e^{2\pi i n x} \end{aligned}$$

- is **well-defined** because the trigonometric series uniformly converges

$$\left\| \sum_{n\in\mathbb{Z}} e^{inh} x_n \right\|_\infty \leq \sum_{n\in\mathbb{Z}} |x_n| = \| x \|_1$$

- is **continuous** again because of the above inequality

$$\left\| \sum_{n\in\mathbb{Z}} e^{inh} x_n \right\|_\infty \leq \| x \|_1$$

  and also has the operator norm ≤ 1.
- has **operator norm** $= 1$ as

$$(0, 0, \dots, \underbrace{1}_{n=1}, 0, 0, 0 \dots) \mapsto 1$$

  and both are of norm 1.

We call the image

Definition. The Weiner algebra on the circle

$$\check{l}^1(S^1) := \left\{ f \in \mathcal{C}(S^1) \mid \hat{f} \in l^1(\mathbb{Z}) \right\}$$

The Fourier transform and *Fourier summation* above are inverses

$$(l^1(\mathbb{Z}), \| \|_1) \cong_{\mathsf{TopVec}_{\mathbb{C}}} (\check{l}^1(S^1), \| \|_\infty)$$

but we may put push the norm on l^1, then

$$(l^1(\mathbb{Z}), \| \|_1) \cong_{\mathsf{Ban}_{\mathbb{C}}} (\check{l}^1(S^1), \| \|_{l^1})$$

$\mathcal{C}^2(S^1) \subset \check{l}^1(S^1)$

- Let $f \in \mathcal{C}^2(S^1)$ or more generally, $f \in \mathcal{C}^1[0,1]$ with $f' \in L^1[0,1]$ and $f(0) = f(1), f'(0) = f'(1)$.
  - From
    - Let $f \in \mathcal{C}^{\mathsf{A}}[0,1]$ so that its differentiable almost everyehere and $f' \in L^1[0,1]$.
      - Then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}\widehat{f}(n) &= \int_{x\in[0,1]} f(x) e^{-2\pi i n x} \\ &= \int_{x\in[0,1]} f(x) \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{-2\pi i n}\frac{\mathrm{d}}{\mathrm{d}x} e^{-2\pi i n x}\right) \\ &= f(0) - f(1) - \frac{1}{-2\pi i n}\int_{x\in[0,1]} f'(x) e^{-2\pi i n x}\end{aligned}$$

      - If $f(0) = f(1)$, that is, $f \in \mathcal{C}^{\mathsf{A}}(S^1)$, then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\widehat{f'}(n) = (2\pi i) n \hat{f}(n)$$

    we know

$$\hat{f}(n) = \frac{\widehat{f''}(n)}{(2\pi i n)^2}$$

  - Further, by
    - For $f \in L^1[0,1]$, the Fourier coefficients of f

$$\hat{f}(n) := \int_{x\in[0,1]} f(x) e^{-2\pi i n x}$$

    is bounded

$$\begin{aligned}\left|\hat{f}(n)\right| &\le \left|\int_{x\in[0,1]} f(x) e^{-2\pi i n x}\right| \\ &\le \int_{x\in[0,1]} |f(x)| \\ &= \|f\|_1\end{aligned}$$

    by its L^1-norm.

we have for $n \in \mathbb{Z} \setminus \{0\}$

$$\left|\hat{f}(n)\right| \leq \frac{\|f''\|_1}{(2\pi i)^2} \frac{1}{n^2}$$

- Thus

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| \leq \left|\hat{f}(0)\right| + \frac{\|f''\|_1}{(2\pi i)^2} \underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}}_{2\frac{\pi^2}{6}}$$

- Thus Fourier summation of f converges *absolutely*.

$\mathcal{C}^1(S^1) \subset \check{l}^1(S^1)$

- Let $f \in \mathcal{C}[0,1]$ with $f' \in L^1[0,1]$ and

$$f(0) = f(1)$$

  - Consider

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| &= \left|\hat{f}(0)\right| + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(|n| \left|\frac{\hat{f}(n)}{n}\right|\right) \\ &\leq \left|\hat{f}(0)\right| + \sqrt{\sum_n \frac{1}{|n|^2}} \sqrt{\sum_n \left|n^2 \hat{f}(n)\right|^2}\end{aligned}$$

  - From
    - Let $f \in \mathcal{C}^{\mathsf{A}}[0,1]$ so that its differentiable almost everyehere and $f' \in L^1[0,1]$.
      - Then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\begin{aligned}\widehat{f}(n) &= \int_{x\in[0,1]} f(x) e^{-2\pi i n x} \\ &= \int_{x\in[0,1]} f(x) \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{-2\pi i n} \frac{\mathrm{d}}{\mathrm{d}x} e^{-2\pi i n x}\right) \\ &= f(0) - f(1) - \frac{1}{-2\pi i n} \int_{x\in[0,1]} f'(x) e^{-2\pi i n x}\end{aligned}$$

      - If $f(0) = f(1)$, that is, $f \in \mathcal{C}^{\mathsf{A}}(S^1)$, then for all $n \in \mathbb{Z} \setminus \{0\}$ we have

$$\widehat{f'}(n) = (2\pi i) n \hat{f}(n)$$

    we know

$$\widehat{f'}(n) = (2\pi i n)^2 \hat{f}(n)$$

  - By

(Parseval's and Riesz-Fischer theorems) Let $f \in L^2[0,1]$. Then the partial Fourier summation converges to f almost everywhere and in L^2

$$S_N f \xrightarrow{N\to\infty} f \qquad \text{ae and } L^2$$

and the Fourier transform is a $\mathbb{C}$-linear bijection

$$\widehat{\ } : L^2[0,1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$$

which satisfies $\|f\|_2 = \left\|\hat{f}\right\|_2$ (is an isometry).

trignometric series for an l^2 sequence

- Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$ and consider

$$T_N(x) :\equiv \sum_{-N\le n\le N} c_n e^{2\pi i n x}$$

  - The series satisfies

$$\|T_N - T_M\|_2^2 = \sum_{M<m\le N} |c_m|^2$$

    and thus $\{T_N\}$ is a Cauchy sequence in $L^2[0,1]$.
  - By completeness of $L^2[0,1]$, there is a limit

$$T := \lim_{N\to\infty}^{L^2} T_N$$

    that means

$$\|T - T_N\|_2 \xrightarrow{N\to\infty} 0$$

  - Fix $n \in \mathbb{Z}$. For $N > |n|$ we have

$$\begin{aligned}\hat{T}(n) - \hat{T}_N(n) &= \int_{[0,1]} (T - T_N) e_n \\ \implies \left|\hat{T}(n) - c_n\right| &\le \|T - T_N\|_1 \\ &\le \|T - T_N\|_2 \\ &\xrightarrow{N\to\infty} 0\end{aligned}$$

  - Therefore,

$$\hat{T}(n) \equiv c_n$$

Hence,

Proposition: Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$. Then

$$\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right) \in L^2[0,1]$$

and it's Fourier transform is c itself

$$\widehat{\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right)}(n) \equiv c_n$$

that is

$$\sum_{n\in\mathbb{Z}} (\ )_n e^{2\pi i n x} : l^2(\mathbb{Z},\mathbb{C}) \to L^2[0,1]$$

when composed with the Fourier transform is $\mathsf{Id}_{l^2(\mathbb{Z},\mathbb{C})}$.

Fourier transform of $L^2[0,1]$

- Let $f \in L^2[0,1]$ and consider a trignometric polynomial

$$\sum_{-N\le n\le N} b_n e^{-2\pi i n\theta}$$

- Then

$$\begin{aligned}
&\left\| f - \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\|_2^2 \\
=&\|f\|_2^2 - \left\langle f, \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\rangle \\
&- \left\langle \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta}, f \right\rangle + \left\| \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\|_2^2 \\
=&\|f\|_2^2 - \sum_{-N\le n\le N} \hat{f}(n)\overline{b_n} - \sum_{-N\le n\le N} \overline{\hat{f}(n)} b_n + \sum_{-N\le n\le N} |b_n|^2 \\
=& \sum_{-N\le n\le N} \left|b_n - \hat{f}(n)\right|^2 + \|f\|_2^2 - \sum_{-N\le n\le N} \left|\hat{f}(n)\right|^2
\end{aligned}$$

- Then the trignometric polynomial for which the *mean square error* $\left\| f - \sum_{-N\le n\le N} b_n e^{-2\pi i n\theta} \right\|_2^2$ is mimum must be when

$$b_n \equiv \hat{f}(n)$$

And then

$$\left\| f - \sum_{-N\leq n\leq N} \hat{f}(n)e^{-2\pi i n\theta} \right\|_2^2 = \|f\|_2^2 - \sum_{-N\leq n\leq N} \left|\hat{f}(n)\right|^2$$

- In particular, because $\| \|_2 \geq 0$ we have

$$\sum_{-N\leq n\leq N} \left|\hat{f}(n)\right|^2 \leq \|f\|_2^2$$

for every $N \geq 0$.

- This implies the sum

$$\left\|\hat{f}\right\|_2^2 = \sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right|^2$$

converges and **(Bessel's inequality)**

$$\left\|\hat{f}\right\|_2^2 \leq \|f\|_2^2$$

- As

$$\hat{f} \in l^2(\mathbb{Z}, \mathbb{C})$$

for $c_n \equiv \hat{f}(n)$, from

> **Proposition:** Let $c_n \in l^2(\mathbb{Z}, \mathbb{C})$. Then
>
> $$\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right) \in L^2[0,1]$$
>
> and it's Fourier transform is c itself
>
> $$\widehat{\left(\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}\right)}(n) \equiv c_n$$
>
> that is
>
> $$\sum_{n\in\mathbb{Z}} (\ )_n e^{2\pi i n x} : l^2(\mathbb{Z}, \mathbb{C}) \to L^2[0,1]$$
>
> when composed with the Fourier transform is $\mathbf{Id}_{l^2(\mathbb{Z},\mathbb{C})}$.

we conclude

$$\widehat{\left(\sum_{m\in\mathbb{Z}} \hat{f}(m)e_m\right)}(n) \equiv \hat{f}(n)$$

- Therefore,

$$\left(\sum_{m\in\mathbb{Z}} \hat{f}(m)e_m\right) \equiv f \qquad \text{ae on } [0,1]$$

and thus

$$S_N f = \sum_{-N\le m\le N} \hat{f}(m)e_m \xrightarrow{N\to\infty} f \qquad \text{ae and } L^2$$

- In particular, taking $\| \|_2$ of the converging sequence implies

$$\sum_{-N\le n\le N} \left|\hat{f}(n)\right| \xrightarrow{N\to\infty} \left\|\hat{f}\right\|_2^2 = \|f\|_2^2$$

∎

we know

$$\|f'\|_2 = \left\|\widehat{f'}\right\|_2 = |2\pi i|\sqrt{\sum_{n\in\mathbb{Z}} \left|n\hat{f}(n)\right|^2}$$

- Therefore,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| &\le \left|\hat{f}(0)\right| + \sqrt{\sum_{n\neq 0} \frac{1}{|n|^2}}\sqrt{\sum_{n\in\mathbb{Z}} \left|n^2\hat{f}(n)\right|^2} \\ &\le \left|\hat{f}(0)\right| + \sqrt{\frac{2\pi^2}{6}}\left\|\widehat{f'}\right\|_2\end{aligned}$$

- In general,

$$\mathcal{C}^\alpha(S^1) \subset \check{l}^1(S^1)$$

for $\alpha > \frac{1}{2}$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Fourier
      - S1 abs Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$

And it has 10 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Fourier
      - L2 bdint Fourier transform on $L^2(-A, A) \leq L^2(\mathbb{R})$
      - L2 Rpos to Hardy2 upper 1 $\widehat{\ } : L^2(0, \infty) \cong_{\mathsf{Hilb}} \mathscr{O}^2(H^2_{\mathrm{U}})$
      - Rn Fourier transform on $\mathbb{R}^n$
      - S1 Fourier transform on S^1, Fourier series on $[0, 1]$
      - S1 abs Functions on S^1 with absolutely converging Fourier series, $\check{l}^1(S^1)$
      - S1 dist Fourier transform of distributions on S^1
      - S1 L1toC0 $\widehat{\ } : L^1[0, 1] \to \mathcal{C}_0(\mathbb{Z}, \mathbb{C})$
      - S1 L2tol2 $\widehat{\ } : L^2[0, 1] \cong_{\mathsf{Hilb}} l^2(\mathbb{Z}, \mathbb{C})$
      - subsets Fourier transform of measurable subsets
      - unsharp Unsharpness principles