Info

This note found here
as a part of a collection
is written (completely with human hands) by Rupadarshi Ray,
created on February 19, 2026 2:31:25 PM,
and was last modified on May 24, 2026 6:41:43 PM.

Laplace operator on $\mathbb{R}^n$

Definition. **We have the Laplace operator**

$$\Delta : \mathcal{C}^2 \to \mathcal{C}$$

$$\begin{aligned}\Delta &:= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \\ &= \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial}{\partial r}(-)\right) + \frac{1}{r^2} \Delta_{S^{n-1}} \\ &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}}\end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial r}(r^{n-1} v'(r)) = 0 \\ \implies & r^{n-1} v'(r) = a \\ \implies & v'(r) = \frac{a}{r^{n-1}} \\ \implies & v(r) = \begin{cases} a \log r + b & n = 2 \\ \frac{a}{r^{n-2}} + b & n \geq 3 \end{cases}\end{aligned}$$

Definition. **The function**

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

is the fundamental solution of Laplace's equation $\Delta u = 0$ on $\mathbb{R}^n$.

Proposition: For

Definition. **The function**

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$
$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

is the fundamental solution of Laplace's equation $\Delta u = 0$ on $\mathbb{R}^n$.

we have

$$|\mathfrak{D}_x \Phi| \lesssim \frac{1}{|x|^{n-1}} \text{ for } x \neq 0$$
$$\left|\mathfrak{D}_x^2 \Phi\right| \lesssim \frac{1}{|x|^{n}} \text{ for } x \neq 0$$

convolution with fundamental solution solves Poisson's equation

Let $f \in \mathcal{C}_c^2(\mathbb{R}^n)$ then the convolution with the fundamental solution

Definition. **The function**

$$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$$
$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

is the fundamental solution of Laplace's equation $\Delta u = 0$ on $\mathbb{R}^n$.

$$\Phi * f : \mathbb{R}^n \to \mathbb{R}$$
$$x \mapsto \int_{y \in \mathbb{R}^n} \Phi(x-y) f(y)$$

is $\Phi * f \in \mathcal{C}^2(\mathbb{R}^n)$ and solves the Poisson's equation $-\Delta u = f$ in $\mathbb{R}^n$.

Let $f \in \mathcal{C}_c^2(\mathbb{R}^n)$ and φ be a fundamental solution of Δ on $\mathbb{R}^n$. Then for

$$u := \varphi * f$$

we have

$$
\begin{aligned}
\Delta u(x) &= \int_{\mathbb{R}^n} \varphi(y)\Delta f(x-y)\mathrm{d}y \\
&= \underbrace{\int_{B_\epsilon(0)} \varphi(y)\Delta f(x-y)\mathrm{d}y}_{\xrightarrow{\epsilon\to 0} 0} + \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \varphi(y)\Delta f(x-y)\mathrm{d}y \\
&\overset{\epsilon\to 0}{=\!\!\longrightarrow} 0 - \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \operatorname{grad}(\varphi)\cdot\operatorname{grad}(f)(x-y)\mathrm{d}y \\
&+ \int_{\partial B_\epsilon(0)} \varphi(y)\underbrace{\frac{\partial f}{\partial \hat{n}}(x-y)}\mathrm{d}y
\end{aligned}
$$

- Now

$$
\int_{\partial B_\epsilon(0)} |\varphi(y)|\mathrm{d}y \le \begin{cases} c\,|\log(\epsilon)|\epsilon \\ c\epsilon \end{cases} \xrightarrow{\epsilon\to 0} 0
$$

- and

$$
\begin{aligned}
&- \int_{\mathbb{R}^n\setminus B_\epsilon(0)} \operatorname{grad}(\varphi)\cdot\operatorname{grad}(f)(x-y)\mathrm{d}y \\
&= \int_{\mathbb{R}^n\setminus B_\epsilon(0)} f(x-y)\operatorname{grad}(\varphi)(y)\mathrm{d}y \;\to 0 \quad - \int_{\partial} \frac{\partial \varphi}{\partial \hat{n}} f(x-y)\mathrm{d}y \\
&= -\{ \\
&\overset{\epsilon\to 0}{=\!\!\longrightarrow} -f(x)
\end{aligned}
$$

Mean-value

Let $u \in \mathcal{C}^2(U)$ and is harmonic, then

$$
u(x) = \frac{1}{m(\partial B_r(x))}\int_{\partial B_r(x)} u = \frac{1}{m(B_r(x))}\int_{B_r(x)} u
$$

for each ball $B_r(x) \subseteq U$.

Let

$$
\begin{aligned}
\varphi(r) &:= \frac{1}{m(\partial B_r(x))}\int_{\partial B_r(x)} u \\
&= \frac{1}{m(\partial B_1(0))}\int_{y\in\partial B_1(0)} u(x+ry)
\end{aligned}
$$

- Then

$$\varphi'(r) = \frac{1}{m(\partial B_1(0))} \int_{y \in \partial B_1(0)} \frac{\partial u}{\partial \hat{n}}(y)$$
$$= \frac{1}{m(\partial B_1(0))} \int_{y \in \partial B_1(0)} \cancelto{0}{\Delta u}$$
$$= 0$$

Estimates on derivatives

Let u is harmonic on $U \subseteq \mathbb{R}^n$. Then

$$\left|\mathfrak{D}_p^\alpha u\right| \leq \frac{c(k)}{r^{n+k}} \| u \|_{L^1(B_r(x_0))}$$

Green's function

$$\Delta G(-, y) = \delta_x$$
$$G|_{\partial U} = 0$$

Current note has 1 direct children and 1 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Laplace Laplace operator on $\mathbb{R}^n$
      - upper half Laplace operator on (flat) upper half space

And it has 36 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - 1Hol Holomorphic functions on spaces over $\mathbb{C}$ of dimension 1
    - circle packing Circle packing on $\mathbb{R}^2$
    - circle packing to Riemann map Circle packing converges to the Riemann biholomorphism
    - Cn conn open bounded Bounded connected open subsets of $\mathbb{C}^n$
    - Cn conn open circular Connected circular open subsets of $\mathbb{C}^n$
    - cont Continuous functions on $\mathbb{R}^d$

- cube dyadic Dyadic cubes
- curves Curves
- derivative Differentiable functions
- forms Differential forms on $\mathbb{R}^n$
- Fourier-Wigner Fourier-Wigner transform
- harmonic composed conformal Harmonic functions composed with conformal maps
- Hilbert Hilbert transform
- hol harmonic disk-circle Fourier-Cauchy-Poisson correspondence of holomorphic and harmonic functions on the unit disk and their boundary values
- Hol sets Holomorphic subsets of $\mathbb{C}^n$
- hypersurf 2n reg Regular hypersurfaces in $\mathbb{R}^{2n}$
- hypersurf or Orientable hypersurfaces in $\mathbb{R}^n$
- KG

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2 + m^2$$

- Laplace Laplace operator on $\mathbb{R}^n$
- Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
- Lmeas bd of open Lebesgue measure of boundary of open sets in $\mathbb{R}^n$
- met density Metric density of subsets of $\mathbb{R}^n$
- Mobius n-sphere Mobius endomorphisms
- monotone Monotone functions on $\mathbb{R}$
- periodic int Cauchy Cauchy integral of periodic functions
- poly int Polygons with integer vertices
- R 2 open smooth End Open smooth maps $U \subseteq \mathbb{R}^2 \to \mathbb{C}$
- R n discrete subg Discrete subgroups of $\mathbb{R}^n$
- R n discrete subg cocpt Discrete cocompact subgroups of $\mathbb{R}^n$, flat tori
- RC ramified germs Ramified germs of smooth and holomorphic functions
- Rn open Open subsets of $\mathbb{R}^n$
- Rn open Riem Open subsets of $\mathbb{R}^n$ equipped with the flat metric
- smooth quasi-analytic Quasi-analytic smooth functions on $\mathbb{R}$
- star shaped Star-shaped subsets of $\mathbb{R}^n$
- Vec ODEs in $\mathbb{R}^n$ $\leftrightarrow$ Vector fields in $\mathbb{R}^n$
- wave

$$\partial_t^2 + \sum_{i=1}^{n} v_i^2 \partial_{x_i}^2$$