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Laplace operator on (flat) upper half space

$$H^n := \mathbb{R}^{n-1} \times (0, \infty)$$

Definition. **The Poisson kernel for flat upper half space is**

$$H^n_{\mathrm{U}} \times \mathbb{R}^{n-1} \to \mathbb{R}$$
$$K(x,y) := \frac{2x_n}{n m_{S^{n-1}}(S^n)} \frac{1}{|x-y|^n}$$

Proposition:

- $K(-,y)$ is harmonic for every $y \in \mathbb{R}^{n-1}$
- $$\int_{\mathbb{R}^{n-1}} K(x,y) \mathrm{d}m_{\mathbb{R}^{n-1}} = 1$$

Therefore from Green's representation we have

$$\begin{cases} \Delta u = 0 & \text{on } H^n_{\mathrm{U}} \\ u = g & \text{on } \mathbb{R}^{n-1} \end{cases} \implies \int_{y \in \mathbb{R}^{n-1}} K(x,y) g(x) \mathrm{d}m_{\mathbb{R}^{n-1}}$$

Now, we can solve the boundary value problem:

Let $u \in \mathcal{C}_b(\mathbb{R}^{n-1})$. Then

$$u(x) := \int_{y\in\mathbb{R}^{n-1}} g(y)K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}}$$

gives $u \in \mathcal{C}_b^\infty(H_U^n)$ such that

$$\begin{cases}\Delta u = 0 & \text{on } H_U^n \\ \lim_{H_U^n \ni x \to x_0} u(x) = g(x_0) & \text{for each } x_0 \in \mathbb{R}^{n-1}\end{cases}$$

As $K \in \mathcal{C}^\infty$ we have $u \in \mathcal{C}^\infty$

- $$\begin{aligned}|u(x) - g(x_0)| &= \left|\int_{y\in\mathbb{R}^{n-1}} (g(y) - g(x_0))K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}}\right| \\ &\le \int_{y\in\mathbb{R}^{n-1}} |g(y) - g(x_0)|K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}} \\ &\le \int_{y\in\mathbb{R}^{n-1}\cap B_\delta(x_0)} \underbrace{|g(y) - g(x_0)|}_{\le\epsilon}K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}} \\ &\quad + \int_{y\in\mathbb{R}^{n-1}\setminus B_\delta(x_0)} |g(y) - g(x_0)|K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}}\end{aligned}$$
    - Consider $x \in B_{\delta/2}(x_0)$ then

$$\begin{aligned}|y - x_0| &\le |y - x| + \underbrace{|x - x_0|}_{\le\delta/2} \\ \implies |y - x_0| &< 2\,|y - x|\end{aligned}$$

    - $$\begin{aligned}&\int_{y\in\mathbb{R}^{n-1}\setminus B_\delta(x_0)} |g(y) - g(x_0)|K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}} \\ \le& 2\|g\|_\infty \int_{y\in\mathbb{R}^{n-1}\setminus B_\delta(x_0)} K(x,y)\mathrm{d}m_{\mathbb{R}^{n-1}} \\ <& 2\|g\|_\infty \frac{2x_n}{n m_{S^{n-1}}(S^{n-1})} \int \frac{1}{|y - x_0|^n} \\ &\xrightarrow{y\to x_0 \implies x_n \to 0} 0\end{aligned}$$

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    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Laplace Laplace operator on $\mathbb{R}^n$
            - upper half Laplace operator on (flat) upper half space

And it has 1 siblings.

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