Info

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is written (completely with human hands) by Rupadarshi Ray,
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Calderon-Zygmund decomposition

dyadic decomposition of cubes and $\mathbb{R}^d$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $R \subseteq \mathbb{R}^d$ be a cube. Then for every $t \in \mathbb{R}$

$$\frac{1}{m(R)} \int_R |f| \geq t$$

there are countably many disjoint dyadic subcubes $\{Q_j\}$ of R such that

$$2^n t \geq \frac{1}{m(Q_j)} \int_{Q_j} |f| > t$$

and

$$|f| \leq t \text{ on } Q \setminus \bigcup_{j \geq 1} Q_j$$

Let $f \in L^1(\mathbb{R}^d)$. Then for every $t \in \mathbb{R}$ there are countably many disjoint dyadic cubes $\{Q_j\}$ in $\mathbb{R}^d$ such that

$$2^n t \geq \frac{1}{m(Q_j)} \int_{Q_j} |f| > t$$

and

$$|f| \leq t \text{ on } \mathbb{R}^d \setminus \bigcup_{j \geq 1} Q_j$$

(Calderson-Zygmund decomposition) Let $f \in L^1(\mathbb{R}^n)$ given $\lambda > 0$ there is a open Ω such that

$$f \leq \lambda \text{ ae on } \Omega^c$$

and

$$\Omega = \bigcup_k Q_k$$

where the dyadic cubes Q_k-s are almost disjoint mutually, and

$$\lambda < \frac{1}{m(Q_k)} \int_{Q_k} f \leq 2^n \lambda$$

decomposition into a sum of bounded and unbounded oscillatory *parts* with zero-mean

Let $f \in L^1(\mathbb{R}^d)$ and $\alpha > 0$. Then there exists a collection $\{Q_j\}$ of disjoint cubes such that

$$\sum_j m(Q_j) \leq \frac{1}{\alpha} \|f\|_1$$

and there exists $g \in L^1 \cap L^\infty(\mathbb{R}^n), b_j \in L^1(Q_j)$ such that

$$f \equiv g + \sum_j b_j$$

where

$$\|g\|_\infty \leq 2^d \alpha$$
$$\|g\|_1 \leq \|f\|_1$$
$$g|_{Q_j} \equiv \frac{1}{m(Q_j)} \int_{Q_j} f$$

and

$$\int_{Q_j} b_j = 0$$
$$\frac{1}{m(Q_j)} \|b_j\|_1 \leq 2^{d+1} \alpha$$

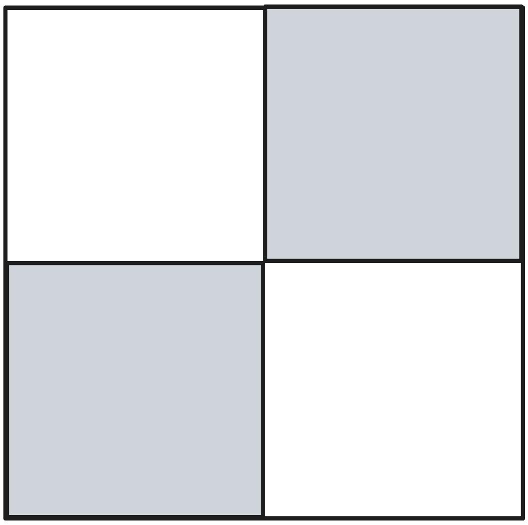
Pick integer $N > 0$ such that

$$\|f\|_1 \leq \frac{\alpha}{2^{dN}}$$

- **We divide $\mathbb{R}^d$ into Ω_N.**

- Consider

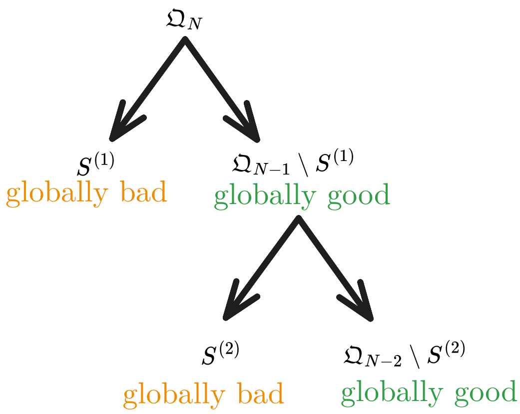
$$S^{(1)} := \left\{ Q \in \mathfrak{Q}_{N-1} \middle| \frac{1}{m(Q)} \int_Q |f| > \alpha \right\}$$



- Cubes in $S^{(1)}$ are where f is globally bad. We consider all cubes in $\mathfrak{Q}_{N-1} \setminus S^{(1)}$ where f behaves globally good. We subdivide each into 2^n subcubes. Then we look that the ones where f is globally bad, and put them in $S^{(2)}$.

- Consider

$$S^{(2)} := \left\{ Q \in \mathfrak{Q}_{N-2}, Q \subseteq \bigcup \mathfrak{Q}_{N-1} \setminus S^{(1)} \middle| \frac{1}{m(Q)} \int_Q |f| > \alpha \right\}$$



- Continuing

$$S^{(k)} := \left\{ Q \in \mathfrak{Q}_{N-k}, Q \subseteq \bigcup \mathfrak{Q}_{N-k+1} \setminus S^{(k-1)} \middle| \frac{1}{m(Q)} \int_Q |f| > \alpha \right\}$$

- ...
- If $\bigcup_{m\geq 1} S^{(m)} = \emptyset$ then set $b :\equiv 0$ and $g :\equiv f$.

- Otherwise, we consider the countable collection of cubes

$$\{Q_j\} := \bigcup_{m\geq 1} S^{(m)}$$

which are all disjoint by construction.

- **Definition.**

$$b_j := \left(f - \frac{1}{m(Q_j)} \int_{Q_j} f\right) 1_{Q_j}$$

- **Definition.**

$$g := f 1_{\mathbb{R}^n \setminus \bigcup_j Q_j} + \sum_j \left(\frac{1}{m(Q_j)} \int_{Q_j} f\right) 1_{Q_j}$$

- By construction $f \equiv g + \sum_j b_j$.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
      - decom CZ Calderon-Zygmund decomposition

And it has 15 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
      - BMO Functions with uniformly bounded mean oscillations on cubes
      - decom bd Chebyshev's inequality
      - decom CZ Calderon-Zygmund decomposition
      - density Lebesgue density of measurable sets
      - DMO Functions with uniformly bounded mean oscillations on dyadic cubes
      - End int Volterra operator $\int_{[0,-]} : L^1_{\text{loc}}[0,1) \to \mathcal{C}[0,1)$
      - f Measurable functions on $\mathbb{R}^n$
      - f quant (ϵ, n)-measurable function
      - int Integrability and integral of measurable functions on $\mathbb{R}^n$

- int HL Hardy-Littlewood maximal functions of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$
- int mean Lebesgue averaging and differentiation
- int monotone Integrals of a monotonically converging sequence of functions
- int undergraph Lebesgue integral from measure of undergraph
- Lorentz $L^{p,q}$
- unit mass $\mathrm{E}([0,1]) = \mathrm{E}(0,1), \mathrm{E}([-\pi,\pi]) = \mathrm{E}(S^1)$