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Integrability and integral of measurable functions on $\mathbb{R}^n$

Definition. Integrability and integral of measurable functions on $\mathbb{R}^n$

Let

$$f : E \subseteq \mathbb{R}^d \to \bar{\mathbb{R}}$$

be a measurable function

- and a simple function $f = \sum a_k \chi_{E_k}$

$$\int f := \sum a_k m(E_k)$$

- and for a bounded function on a set of finite measure

$$\varphi_n \to f \implies \int f := \lim_{n\to\infty} \int \varphi_n$$

- and $f \geq 0$ then f is integrable if

$$\int f := \sup_{0\leq g\leq f} \int g$$

  exists where the supremum is taken over all bounded functions g supported on a set of finite measure
- and in general f is **Lebesgue integrable** if $|f|$ is *integrable* by the previous criterion and

$$\int f := \int f^+ - \int f^-$$

  where $f^+(x) := \max\{f(x), 0\}, f^-(x) := \max\{-f(x), 0\}$

Summary of integrability of measurable functions

	supported on finite meas	support has ∞ measure
bounded	any measurable function is integrable!	very easy examples like $\chi_{[0,\infty)}$ on $\mathbb{R}$ is not integrable, $\int_{\mathbb{R}} \sin x/x$ is also not integrable
unbounded, $f \geq 0$	$\int_{(0,1]} \frac{1}{x}$ does not exist but $$\int_{(0,1]} \frac{1}{\sqrt{x}}$$ does	
unbounded, possibly negative in places		

bounded and supported on finite meas

(Integral of bounded functions supported on finite measure exists) Let f be a bounded function on a set $E \subseteq \mathbb{R}^d$ of finite measure. If (φ_n) is any sequence of simple functions bounded by M supported on E with

$$\varphi_n \to f \text{ aeon } E$$

then

$$\lim_{n\to\infty} \int_E \varphi_n$$

exists and

$$f = 0 \text{ aeon } E \implies \lim_{n\to\infty} \int_E \varphi_n = 0$$

It is also unique, just by the statement!

(Bounded convergence) Let (f_n) be meas, bounded by M, supported on finite meas set E and $f_n \to f$ aeon E

$$\implies$$

f is measurable, bounded, supported on E and

$$\int_E f_n \to \int_E f$$

$f \geq 0$

- If $f > 0$
  - $$\int f = \sup_{0 \le g \le f} \int g$$
  - and $\int f = 0$ then $f = 0$ almost everywhere.

Example

Let

$$f_n(x) := n\chi_{[0,1/n]}$$

then $f_n(x) \to 0$ for all x but

$$\int_{\mathbb{R}} f_n = 1$$

for all n.

(Fatou) Let $f_n \ge 0$ be a sequence of measurable functions

$$\int \lim_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$$

(Monotone convergence) Let $f_n : \mathbb{R}^d \to [0, \infty)$ be a sequence of measurable functions and $f_n \underset{n\to\infty}{\nearrow} f$ aeon $\mathbb{R}^d$. Then

$$\lim_{n\to\infty} \int f_n = \int f$$

If $a_k(x) \ge 0$ and measble then

$$\int \sum_{k=1}^{n} a_k(x) \mathrm{d}x = \sum_{k=1}^{n} \int a_k(x) \mathrm{d}x$$

If $\sum_{k=1}^{\infty} \int a_k < \infty$ then $\sum_{k=1}^{\infty} a_k(x)$ is convergent ae.

Suppose $\{E_k\}$ be a collection of subsets with $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then the set of points that the set of points that belong to infinitely many sets E_k

$$m(\{x : x \in E_k \text{ for infinitely many } k\}) = 0$$

has measure zero.

■

Take $a_k(x) = \chi_{E_k(x)}$ and use

If $a_k(x) \geq 0$ and measble then

$$\int \sum_{k=1}^{n} a_k(x)\mathrm{d}x = \sum_{k=1}^{n} \int a_k(x)\mathrm{d}x$$

If $\sum_{k=1}^{\infty} \int a_k < \infty$ then $\sum_{k=1}^{\infty} a_k(x)$ is convergent ae.

- $f_n := \sum_{k=1}^{n} a_k(x)$ and $f(x) := \sum_{k=1}^{\infty} a_k(x)$ then $f_n \uparrow f$ and

$$\int f_n = \sum_{k=1}^{n} \int a_k(x)\mathrm{d}x$$

- By MCT,

$$\int \sum_{k=1}^{n} a_k(x)\mathrm{d}x = \sum_{k=1}^{n} \int a_k(x)\mathrm{d}x$$

Definition.

$$(L^1(E), +, \ , \| \ \|_1)$$

The integral of Lebesgue integrable functions

$$\int : L^1(E) \to \mathbb{R}$$

is linear, additive, monotonic and satisfies triangle inequality.

(Integrable functions on $\mathbb{R}^d$ "vanish at infinity") Let $f \in L^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists a ball $B_{N(\epsilon)} \subseteq \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d \setminus B_{N(\epsilon)}} |f| < \epsilon$$

For any $g \in L^1(\mathbb{R}^d)$ take $f := |g|$. Consider the trunctation

$$f 1_{B_N} \underset{N\to\infty}{\nearrow} f$$

- By

(Monotone convergence) **Let $f_n : \mathbb{R}^d \to [0, \infty)$ be a sequence of measurable functions and $f_n \underset{n\to\infty}{\nearrow} f$ aeon $\mathbb{R}^d$. Then**

$$\lim_{n\to\infty} \int f_n = \int f$$

we have

$$\int_{\mathbb{R}^d} f 1_{B_N} \xrightarrow{N\to\infty} \int_{\mathbb{R}^d} f$$

- This means $\forall \epsilon \exists N(\epsilon)$ such that

$$0 \le \int_{\mathbb{R}^d} \underbrace{(f - f 1_{B_{N(\epsilon)}})}_{f 1_{B^c_{N(\epsilon)}}} < \epsilon$$

$$\implies \int_{B^c_{N(\epsilon)}} f < \epsilon$$

(Absolute continuity of $f dm$) **Let $f \in L^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for every measurable set E such that $m(E) < \delta_\epsilon$**

$$\int_E |f| < \epsilon$$

(Dominated convergence) **Let the sequence of measurable functions $f_n \to f$ ae pointwise and**

$$|f_n(x)| \le g(x)$$

for some $g \in L^1$. Then

$$\int |f_n - f| \xrightarrow{n\to\infty} 0$$

chopping integral into lower dimensions

(Fubini's theorem) Suppose $f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Then for almost every $y \in \mathbb{R}^{d_2}$

$$f(-, y) \in L^1(\mathbb{R}^{d_1})$$

and

$$y \mapsto \int_{\mathbb{R}^{d_1}} f(-, y) \in L(\mathbb{R}^{d_2})$$

Moreover,

$$\int_{y \in \mathbb{R}^{d_2}} \left(\int_{x \in \mathbb{R}^{d_1}} f(x, y) \right) = \int_{\mathbb{R}^{d_1 + d_2}} f$$

differentiation

differentiation of the integral

stamp.Rf.Lmeas.int HL

stamp.Rf.Lmeas.int mean

differentiable almost everywhere on $[a, b]$

Definition. **Variation of functions $f : [a, b] \to \mathbb{R}$**

The total variation of $f : [a, b] \to \mathbb{R}$

$$V_{[a,b]}(f) := \sup_{\text{paritions}} \sum_{i=1}^{N} |f(t_j) - f(t_{j-1})|$$

f is of bounded variation if $V_{[a,b]}(f) < \infty$.

Let $V_{[a,b]}(f) < \infty$ consider

$$V^{+}_{[a,b]}(f) := \sup_{\text{partitions of } [a,b]} \sum_{j\,:\,F(t_j) \geq F(t_{j-1})} F(t_j) - F(t_{j-1})$$

and

$$V^{-}_{[a,b]}(f) := \sup_{\text{partitions of } [a,b]} \sum_{j\,:\,F(t_j) \leq F(t_{j-1})} -F(t_j) + F(t_{j-1})$$

$$f(x) = f(a) + V^{+}_{[a,x]}(f) - V^{-}_{[a,x]}(f)$$

where both the functions on right are increasing and are of bounded variation.

$$V = V^{-} + V^{+}$$

$$F : [a,b] \to \mathbb{R}$$

is of *bounded variation*

$$\iff$$

F is the difference of two increasing bounded functions.

$$F : [a,b] \to \mathbb{R}$$

is of *bounded variation*

$$\implies$$

F is differentiable almost everywhere.

Absolutely continuous functions on $[a,b] \leftrightarrow \int_{[a,-]}(L^1[a,b])$

Definition. Absolutely continuous function on $[a,b]$

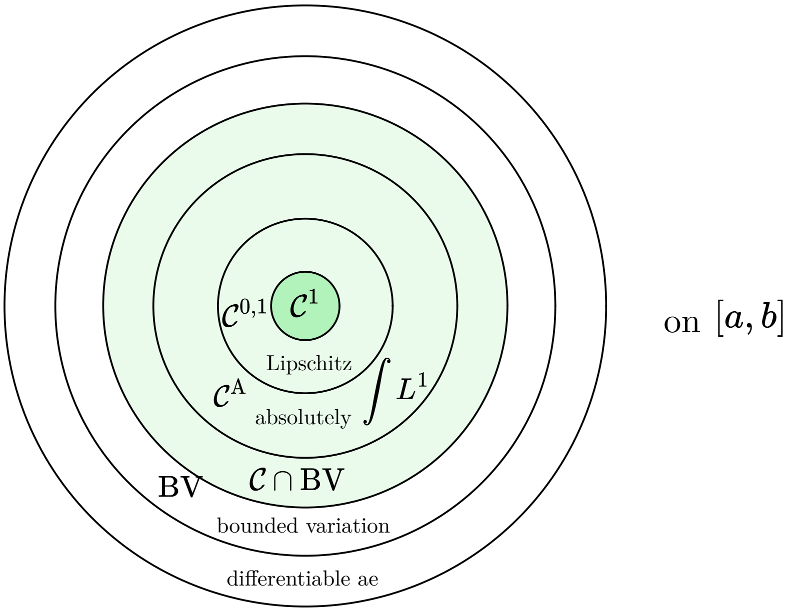
A function F defined on $[a,b]$ is **absolutely continuous** if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ so that for every collection $(a_k, b_k) \subseteq [a,b]$ of disjoint intervals $1 \leq k \leq N$ we have

$$\sum_{k=1}^{n} (b_k - a_k) < \delta(\epsilon) \implies \sum_{k=1}^{n} |F(b_k) - F(a_k)| < \epsilon$$

The space of all absolutely continuous functions on $[a,b]$

$$\mathcal{C}^{\mathsf{A}}[a,b]$$

may equipped with the supremum $\| \|_\infty$ or variation V norms.



$\int_{[a,-]} : L^1[a,b] \to \mathcal{C}^{\mathsf{A}}[a,b]$

- Let $f \in L^1[a,b]$ and consider

$$F(x) := \int_{[a,x]} f$$

- Then

$$\begin{aligned} F(x) - F(y) &= \int_{[a,x]} f - \int_{[a,y]} f \\ &= \int_{[x,y]} f \\ \implies |F(x) - F(y)| &\leq \int_{[x,y]} |f| \end{aligned}$$

- By

(Absolute continuity of $f\mathrm{d}m$) Let $f \in L^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for every measurable set E such that $m(E) < \delta_\epsilon$

$$\int_E |f| < \epsilon$$

we have $\forall \epsilon > 0 \exists \delta(\epsilon) > 0$ such that for every

$$I = \bigsqcup_k (a_k, b_k)$$

such that

$$m(I) = \sum_k (b_k - a_k) < \delta(\epsilon)$$

the integral

$$\int_I |f| < \epsilon$$

- Here,

$$\begin{aligned} |F(b_k) - F(a_k)| &\leq \int_{[a_k,b_k]} |f| \\ \implies \sum_k |F(b_k) - F(a_k)| &\leq \sum_k \int_{[a_k,b_k]} |f| \\ &= \int_I |f| \\ &< \epsilon \end{aligned}$$

$\mathfrak{D} : \mathcal{C}^{\mathsf{A}}[a,b] \to L^1[a,b]$

(Fundamental theorem of analysis, in the almost everywhere sense) A function

$$F : [a, b] \to \mathbb{R}$$

is *absolutely continuous* on $[a, b] \implies F'$ exists m-almost everywhere and is L^1. In that case,

$$F(x) = F(a) + \int_{[a,x]} F' \qquad \text{on } [a, b]$$

Moreover $F' = 0$ m-almost everywhere $\implies F$ is constant m-almost everywhere.
Conversely, for every $f \in L^1[a, b]$ there exists a function F which is differentiable m-almost everywhere and

$$F' = f \text{ ae}$$

and in fact we may take F to be the absolutely continuous function $x \mapsto c_0 + \int_{[a,x]} f$ for any $c_0 \in \mathbb{C}$.

$$x, y \in [a, b] \implies |f(x) - f(y)| \leq K|x - y|$$

- if $K = 0$, f is constant
- if $K > 0$,

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \leq K \sum_{i=1}^{n} |b_i - a_i|$$

Let $f, f' \in L^1(\mathbb{R})$

- Then

$$\lim_{x\to\infty} \underbrace{\int_{[0,x]} f'}_{f(x)-f(0)} = \int_{[0,\infty)} f' =: M < \infty$$

but if $M \neq 0$ then

$$\lim_{x\to\infty} f(x) - f(0) = M$$

which is impossible, so $M=0$ and

$$f(x) \xrightarrow{x\to\infty} 0$$

	L^1	
pointwise	$$\chi_{[n,n+1]} \in L^1, \to 0$$ but $$\int_{\mathbb{R}} \chi_{[n,n+1]} = 1$$	

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
      - int Integrability and integral of measurable functions on $\mathbb{R}^n$

And it has 15 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
      - BMO Functions with uniformly bounded mean oscillations on cubes
      - decom bd Chebyshev's inequality
      - decom CZ Calderon-Zygmund decomposition
      - density Lebesgue density of measurable sets
      - DMO Functions with uniformly bounded mean oscillations on dyadic cubes
      - End int Volterra operator $\int_{[0,-]} : L^1_{\text{loc}}[0,1) \to \mathcal{C}[0,1)$
      - f Measurable functions on $\mathbb{R}^n$
      - f quant (ϵ, n)-measurable function
      - int Integrability and integral of measurable functions on $\mathbb{R}^n$
      - int HL Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$

- int mean Lebesgue averaging and differentiation
- int monotone Integrals of a monotonically converging sequence of functions
- int undergraph Lebesgue integral from measure of undergraph
- Lorentz $L^{p,q}$
- unit mass $\mathrm{E}([0,1]) = \mathrm{E}(0,1), \mathrm{E}([-\pi,\pi]) = \mathrm{E}(S^1)$