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> is written (completely with human hands) by Rupadarshi Ray,
> created on August 2, 2025 5:20:02 PM,
> and was last modified on May 25, 2026 12:10:29 AM.

Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$

Definition. Hardy-Littlewood maximal function of $L^1_{\text{loc}}(\mathbb{R}^n)$

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n, m)$ then

- $$(\mathrm{M}f)(x) := \sup_{\delta > 0} \frac{1}{m(B(x,\delta))} \int_{B(x,\delta)} |f|$$

  is the **centered Hardy-Littlewood maximal function**

- $$(\mathrm{Mu}f)(x) := \sup_{\delta > 0, |x-y| < \delta} \frac{1}{m(B(y,\delta))} \int_{B(y,\delta)} |f|$$

  is the **uncentered** one

- $$\mathrm{M}(f) = \mathrm{M}(|f|) \geq 0$$

  and M is thus positive and semi-linear

- $$\mathrm{M} : L^\infty \to L^\infty$$

  with

  $$\|\mathrm{M}(f)\|_\infty \leq \|f\|_\infty$$

- $$\mathrm{M}(f) \leq \mathrm{Mu}(f) \leq 2^n \mathrm{M}(f)$$

- The linear functional

$$
\begin{aligned}
(\mathrm{Mu}f)(x) &= \sup_{\delta>0} \frac{1}{m(B_1)\delta^n} \int_{y\in R^n} |f(x-y)|\chi_{B_1}\left(\frac{y}{\delta}\right) \\
&= \sup_{\delta>0} |f| * \underbrace{\frac{\chi_{B_1}\left(\frac{y}{\delta}\right)}{\delta^n}}_{\chi_{B_1,\delta}}
\end{aligned}
$$

is thus supremum of convolutions with dilations of the characteristic of the ball.

maximal inequalities

Lemma 2.1.5. *Let $\{B_1, B_2, \ldots, B_k\}$ be a finite collection of open balls in $\mathbf{R}^n$. Then there exists a finite subcollection $\{B_{j_1}, \ldots, B_{j_l}\}$ of pairwise disjoint balls such that*

$$
\sum_{r=1}^{l} |B_{j_r}| \geq 3^{-n} \left| \bigcup_{i=1}^{k} B_i \right| . \tag{2.1.2}
$$

Proof. Let us reindex the balls so that

$$
|B_1| \geq |B_2| \geq \cdots \geq |B_k| .
$$

Let $j_1 = 1$. Having chosen $j_1, j_2, \ldots, j_i$, let j_{i+1} be the least index $s > j_i$ such that $\bigcup_{m=1}^{i} B_{j_m}$ is disjoint from B_s. Since we have a finite number of balls, this process will terminate, say after l steps. We have now selected pairwise disjoint balls $B_{j_1}, \ldots, B_{j_l}$. If some B_m was not selected, that is, $m \notin \{j_1, \ldots, j_l\}$, then B_m must intersect a selected ball B_{j_r} for some $j_r < m$. Then B_m has smaller size than B_{j_r} and we must have $B_m \subsetneq 3B_{j_r}$. This shows that the union of the unselected balls is contained in the union of the triples of the selected balls. Therefore, the union of all balls is contained in the union of the triples of the selected balls. Thus

$$
\left| \bigcup_{i=1}^{k} B_i \right| \leq \left| \bigcup_{r=1}^{l} 3B_{j_r} \right| \leq \sum_{r=1}^{l} |3B_{j_r}| = 3^n \sum_{r=1}^{l} |B_{j_r}| ,
$$

and the required conclusion follows. □

(Hardy–Littlewood maximal inequality) The operators

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map

$$\mathrm{M}, \mathrm{Mu} : L^1 \to L^{1,\infty}$$

with norm $\leq 3^n$ and

$$\mathrm{M}, \mathrm{Mu} : L^p \to L^p$$

with norm $\leq 3^{n/p}\frac{p}{p-1}$. Moreover for $f \in L^1, \alpha > 0$ we have weak-boundedness

$$m(\mathrm{M}f > \alpha) \leq \frac{c(n)}{\alpha} \int_{\mathrm{M}f>\alpha} |f|$$

where $c(n) \leq 3^n$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
      - int HL Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$

And it has 15 siblings.

- stamp stamp
  - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
    - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
      - BMO Functions with uniformly bounded mean oscillations on cubes
      - decom bd Chebyshev's inequality
      - decom CZ Calderon-Zygmund decomposition
      - density Lebesgue density of measurable sets
      - DMO Functions with uniformly bounded mean oscillations on dyadic cubes
      - End int Volterra operator $\int_{[0,-]} : L^1_{\text{loc}}[0,1) \to \mathcal{C}[0,1)$
      - f Measurable functions on $\mathbb{R}^n$
      - f quant (ϵ, n)-measurable function
      - int Integrability and integral of measurable functions on $\mathbb{R}^n$

- int HL Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$
- int mean Lebesgue averaging and differentiation
- int monotone Integrals of a monotonically converging sequence of functions
- int undergraph Lebesgue integral from measure of undergraph
- Lorentz $L^{p,q}$
- unit mass $\mathrm{E}([0,1]) = \mathrm{E}(0,1), \mathrm{E}([-\pi,\pi]) = \mathrm{E}(S^1)$