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Mean and oscillation of integrable functions

average of L^1_{loc} functions

We have

$$\begin{aligned} A_r : L^1_{\text{loc}}(\mathbb{R}^n) &\to \mathcal{C}(\mathbb{R}^n) \\ x &\mapsto \frac{1}{m(B_r(x))} \int_{B_r(x)} f \end{aligned}$$

and

$$(r, x) \mapsto A_r f(x)$$

is a continuous map.

mean oscillation of L^1_{loc} functions

Definition. **Mean oscillation of integrable functions**

Let $g \in L^1_{\text{loc}}(\mathbb{R}^d)$. The **relative oscillation** of f on E about x is the mean of $|g - g(x)|$ on E

$$\frac{1}{m(E)} \int_E |g - g(x)|$$

The **mean oscillation** of f on E is

$$\frac{1}{m(E)} \int_E \left| g - \left(\frac{1}{m(E)} \int_E g \right) \right|$$

Proposition:

$$|A_r f - f|(x) \leq \frac{1}{m(B_r)} \int_{B_r(x)} |g - g(x)|$$

infinitesimal relative oscillation and infinitesimal averaging on balls

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The infinitesimal ($\limsup$) relative oscillation of f on balls about x

$$\limsup_{\delta \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g - g(x)|$$

is 0 for almost all $x \in \mathbb{R}^d$.

(Differentiation of the Lebesgue integral on balls) If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ then

$$\lim_{r \to 0} (A_r f)(x) = f(x) \text{ aeon } x \in \mathbb{R}^d$$

- **Proposition:** These two theorems are equivalent.
    - **Proposition:**

      $$|A_r f - f|(x) \leq \frac{1}{m(B_r)} \int_{B_r(x)} |g - g(x)|$$

    - For each $c \in \mathbb{C}$ we consider $g_c := |f - c|$ and by

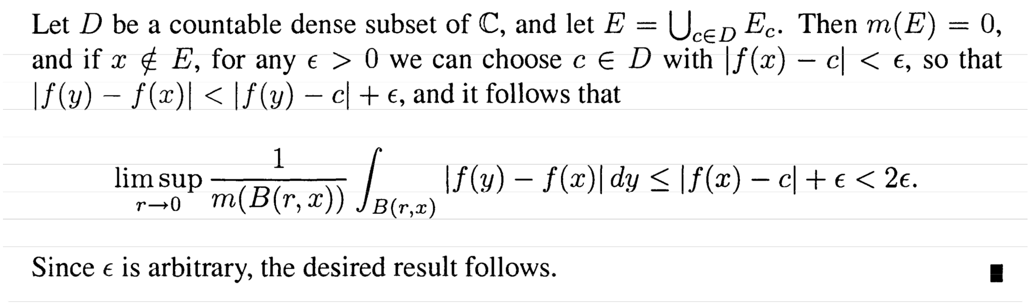
      (Differentiation of the Lebesgue integral on balls) If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ then

      $$\lim_{r \to 0} (A_r f)(x) = f(x) \text{ aeon } x \in \mathbb{R}^d$$

      conclude that

      $$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B_r(x)} |f(y) - c| = |f(x) - c| \text{ on } \mathbb{R}^d \setminus E_c$$

where E_c is a set of Lebesgue measure zero.



Let D be a countable dense subset of $\mathbb{C}$, and let $E = \bigcup_{c \in D} E_c$. Then $m(E) = 0$, and if $x \notin E$, for any $\epsilon > 0$ we can choose $c \in D$ with $|f(x) - c| < \epsilon$, so that $|f(y) - f(x)| < |f(y) - c| + \epsilon$, and it follows that

$$\limsup_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)|\, dy \le |f(x) - c| + \epsilon < 2\epsilon.$$

Since ϵ is arbitrary, the desired result follows. ∎

Definition. **Lebesgue points**

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The points $x \in \mathbb{R}^d$ where the *infinitesimal relative oscillation* of f on balls about x is 0 are called **Lebesgue points** of f and the set of all such points is the **Lebesgue set** of f

$$\text{Leb}(f) := \left\{ x \in \mathbb{R}^d \middle| \lim_{\delta \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g - g(x)| = 0 \right\}$$

Proposition: From

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The infinitesimal ($\limsup$) relative oscillation of f on balls about x

$$\limsup_{\delta \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g - g(x)|$$

is 0 for almost all $x \in \mathbb{R}^d$.

we conclude the Lebesgue set

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is a full-Lebesgue measure.

for continuous functions

For continuous functions, the average is easily obtained.

Let $g \in \mathcal{C} \cap L^1(\mathbb{R}^n)$. Then for every $x \in \mathbb{R}^n, \delta > 0$ there exists $r(\delta, x) > 0$ such that

$$\forall y \in B_{r(\delta,x)}(x), |g(y) - g(x)| < \delta$$

- For all $x \in \mathbb{R}^n, \epsilon > 0$, if $r < r(\delta, x)$ we have $y \in B_r(x) \subseteq B_{r(\delta,x)}$ then

$$\begin{aligned}|(A_r g - g)(x)| &= \frac{1}{m(B_r(x))}\left|\int_{B_r(x)} (g - g(x))\right| \\ &\leq \frac{1}{m(B_r(x))}\int_{B_r(x)} \underbrace{|g - g(x)|}_{<\delta} \\ &< \delta\end{aligned}$$

- Therefore,

$$\lim_{r \to 0}(A_r f)(x) = f(x)$$

  for all $x \in \mathbb{R}^n$.

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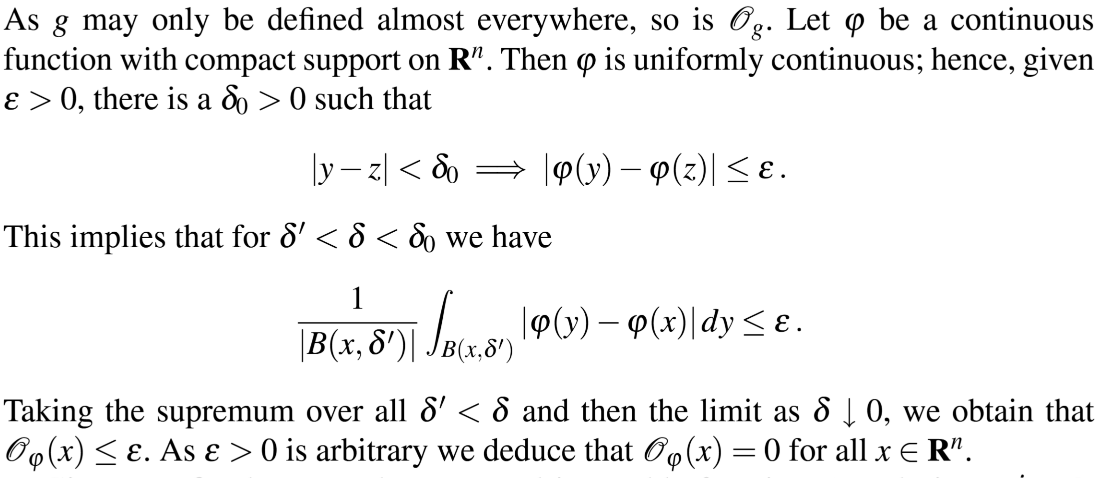

As g may only be defined almost everywhere, so is $\mathscr{O}_g$. Let φ be a continuous function with compact support on $\mathbf{R}^n$. Then φ is uniformly continuous; hence, given $\varepsilon > 0$, there is a $\delta_0 > 0$ such that

$$|y - z| < \delta_0 \implies |\varphi(y) - \varphi(z)| \leq \varepsilon .$$

This implies that for $\delta' < \delta < \delta_0$ we have

$$\frac{1}{|B(x,\delta')|}\int_{B(x,\delta')} |\varphi(y) - \varphi(x)|\, dy \leq \varepsilon .$$

Taking the supremum over all $\delta' < \delta$ and then the limit as $\delta \downarrow 0$, we obtain that $\mathscr{O}_\varphi(x) \leq \varepsilon$. As $\varepsilon > 0$ is arbitrary we deduce that $\mathscr{O}_\varphi(x) = 0$ for all $x \in \mathbf{R}^n$.

for L^1_{loc}

Caution

By

> **(Lusin) Let f be measurable function on $E \subseteq \mathbb{R}^d$ with $m(E) < \infty$. Then for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subset E$ such that $m(E - F_\epsilon) \leq \epsilon$ so that**
>
> $$f\Big|_{F_\epsilon}$$
>
> **is continuous.**

we have a F_ϵ such that f is continuous on F_ϵ. Then by

💡 Let $g \in \mathcal{C} \cap L^1(\mathbb{R}^n)$. Then for every $x \in \mathbb{R}^n, \delta > 0$ there exists $r(\delta, x) > 0$ such that

$$\forall y \in B_{r(\delta,x)}(x), |g(y) - g(x)| < \delta$$

- For all $x \in \mathbb{R}^n, \epsilon > 0$, if $r < r(\delta, x)$ we have $y \in B_r(x) \subseteq B_{r(\delta,x)}$ then

$$\begin{aligned} |(A_r g - g)(x)| &= \frac{1}{m(B_r(x))} \left| \int_{B_r(x)} (g - g(x)) \right| \\ &\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} \underbrace{|g - g(x)|}_{<\delta} \\ &< \delta \end{aligned}$$

- Therefore,

$$\lim_{r \to 0} (A_r f)(x) = f(x)$$

  for all $x \in \mathbb{R}^n$.

we have

$$\left\{ \lim_{r \to 0} (A_r f)(x) \neq f(x) \right\} \subseteq \underbrace{\mathbb{R}^d \setminus F_\epsilon}_{m \leq \epsilon}$$

Therefore for each $\epsilon > 0$, $\lim_{r \to 0}(A_r f)(x) \neq f(x)$ on a set of measure $\leq \epsilon$. Thus $\lim_{r \to 0}(A_r f)(x) = f(x)$ on a full-measure set.

The above argument, however, is incorrect.

The function f when restricted to F_ϵ is continuous $F_\epsilon \to \mathbb{C}$ with respect to the subspace topology on F_ϵ. This means it satisfies

$$\forall x \in F_\epsilon, \epsilon > 0 \exists r(x, \delta) > 0 : \forall y \in B_{r(\delta,x)} \cap F_\epsilon, |g(y) - g(x)| < \delta$$

This does not help us in concluding

$$\forall r < r(\delta, x), \frac{1}{m(B_r)} \int_{B_r(x)} |g - g(x)| < \delta$$

as the integral is on the entire ball $B_r(x)$ and not on the subspace ball $B_r(x) \cap F_\epsilon$

[1]

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
            - int mean Lebesgue averaging and differentiation

And it has 15 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
            - BMO Functions with uniformly bounded mean oscillations on cubes
            - decom bd Chebyshev's inequality
            - decom CZ Calderon-Zygmund decomposition
            - density Lebesgue density of measurable sets
            - DMO Functions with uniformly bounded mean oscillations on dyadic cubes
            - End int Volterra operator $\int_{[0,-]} : L^1_{\text{loc}}[0,1) \to \mathcal{C}[0,1)$
            - f Measurable functions on $\mathbb{R}^n$
            - f quant (ϵ, n)-measurable function
            - int Integrability and integral of measurable functions on $\mathbb{R}^n$
            - int HL Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$
            - int mean Lebesgue averaging and differentiation
            - int monotone Integrals of a monotonically converging sequence of functions
            - int undergraph Lebesgue integral from measure of undergraph
            - Lorentz $L^{p,q}$
            - unit mass $\mathrm{E}([0,1]) = \mathrm{E}(0,1), \mathrm{E}([-\pi,\pi]) = \mathrm{E}(S^1)$

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1. https://math.stackexchange.com/q/4045572/1290493 ↩