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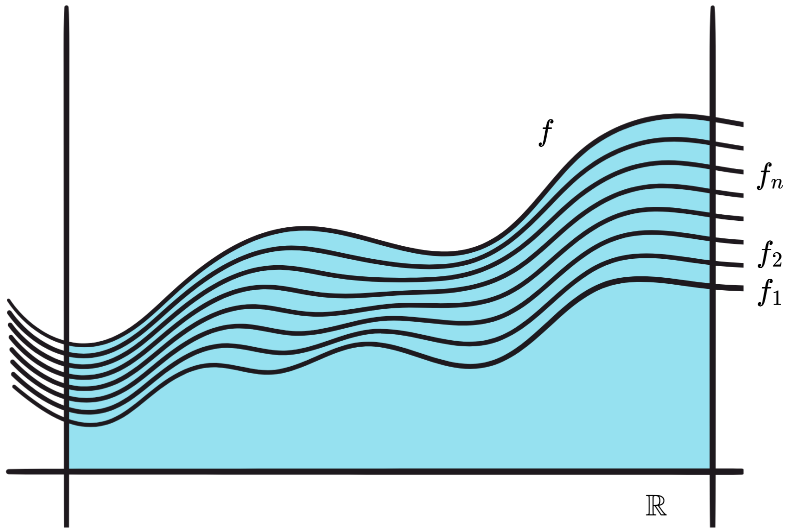
is written (completely with human hands) by Rupadarshi Ray,
created on May 28, 2026 12:08:28 AM,
and was last modified on June 12, 2026 7:54:29 PM.

Integrals of a monotonically converging sequence of functions

(Monotone convergence) Let $f_n : \mathbb{R}^d \to [0, \infty)$ be a sequence of measurable functions and $f_n \underset{n\to\infty}{\nearrow} f$ aeon $\mathbb{R}^d$. Then

$$\lim_{n\to\infty} \int f_n = \int f$$

via Lebesgue integral from measure of undergraph



- We have

$$f_n \uparrow f \implies \bigcup_n \mathcal{U} f_n = \mathcal{U} f$$

- Then by

- Let

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subseteq A_{j+1} \subseteq \dots$$

is an **increasing** sequence of measurable sets.

- Then

$$\begin{aligned}\mu\left(\bigcup_{j\geq 1} A_j\right) &= \mu\left(\bigcup_{j\geq 1}(A_j \setminus (A_1 \cup \cdots \cup A_{j-1}))\right) \\ &= \sum_{j\geq 1} \mu(A_j \setminus (A_1 \cup \cdots \cup A_{j-1})) \\ &= \lim_{n\to\infty} \sum_{1\leq j\leq N} \mu(A_j \setminus (A_1 \cup \cdots \cup A_{j-1})) \\ &= \lim_{n\to\infty} \mu\left(\bigcup_{1\leq j\leq n} A_j \setminus (A_1 \cup \cdots \cup A_{j-1})\right) \\ &= \lim_{n\to\infty} \mu(A_n)\end{aligned}$$

we have

$$m(\mathcal{U}f) = \lim_{n\to\infty} \mathcal{U}f_n$$

truncting at balls $\implies$ integrals vanish at infinity

(Integrable functions on $\mathbb{R}^d$ "vanish at infinity") Let $f \in L^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists a ball $B_{N(\epsilon)} \subseteq \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d \setminus B_{N(\epsilon)}} |f| < \epsilon$$

For any $g \in L^1(\mathbb{R}^d)$ take $f := |g|$. Consider the trunctation

$$f 1_{B_N} \underset{N\to\infty}{\nearrow} f$$

- By

(Monotone convergence) Let $f_n : \mathbb{R}^d \to [0, \infty)$ be a sequence of measurable functions and $f_n \underset{n\to\infty}{\nearrow} f$ aeon $\mathbb{R}^d$. Then

$$\lim_{n\to\infty} \int f_n = \int f$$

we have

$$\int_{\mathbb{R}^d} f 1_{B_N} \xrightarrow{N\to\infty} \int_{\mathbb{R}^d} f$$

- This means $\forall \epsilon \exists N(\epsilon)$ such that

$$0 \leq \int_{\mathbb{R}^d} \underbrace{(f - f1_{B_{N(\epsilon)}})}_{f1_{B^c_{N(\epsilon)}}} < \epsilon$$
$$\implies \int_{B^c_{N(\epsilon)}} f < \epsilon$$

truncating at $\{f \leq N\} \implies$ absolute continuity

(Absolute continuity of $f\mathrm{d}m$) Let $f \in L^1(\mathbb{R}^d)$. Then for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for every measurable set E such that $m(E) < \delta_\epsilon$

$$\int_E |f| < \epsilon$$

For any $g \in L^1(\mathbb{R}^d)$ take $f := |g|$. Consider the truncation

$$f1_{\{f \leq N\}} \underset{n \to \infty}{\nearrow} f$$

- By

  (Monotone convergence) Let $f_n : \mathbb{R}^d \to [0, \infty)$ be a sequence of measurable functions and $f_n \underset{n \to \infty}{\nearrow} f$ aeon $\mathbb{R}^d$. Then

  $$\lim_{n \to \infty} \int f_n = \int f$$

  $\forall \epsilon \exists N(\epsilon)$ such that

$$\int_{\mathbb{R}^d} (f - f1_{\{f \leq N(\epsilon)\}}) < \frac{\epsilon}{2}$$

- Now pick $\delta(\epsilon)$ such that

$$\delta(\epsilon) < \frac{\epsilon}{2N(\epsilon)}$$

- For any measurable E with $m(E) < \delta(\epsilon)$ we have

$$\begin{aligned}\int_E f &= \int_E (f - f1_{\{f \leq N(\epsilon)\}}) + \int_E f1_{\{f \leq N\}} \\ &\leq \int_{\mathbb{R}^d} (f - f1_{\{f \leq N(\epsilon)\}}) + Nm(E) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
            - int monotone Integrals of a monotonically converging sequence of functions

And it has 15 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - Lmeas Lebesgue measurable subsets of and functions on $\mathbb{R}^n, T^n, S^n$
            - BMO Functions with uniformly bounded mean oscillations on cubes
            - decom bd Chebyshev's inequality
            - decom CZ Calderon-Zygmund decomposition
            - density Lebesgue density of measurable sets
            - DMO Functions with uniformly bounded mean oscillations on dyadic cubes
            - End int Volterra operator $\int_{[0,-]} : L^1_{\text{loc}}[0,1) \to \mathcal{C}[0,1)$
            - f Measurable functions on $\mathbb{R}^n$
            - f quant (ϵ, n)-measurable function
            - int Integrability and integral of measurable functions on $\mathbb{R}^n$
            - int HL Hardy-Littlewood maximal functions of $L^1_{\text{loc}}(\mathbb{R}^n)$
            - int mean Lebesgue averaging and differentiation
            - int monotone Integrals of a monotonically converging sequence of functions
            - int undergraph Lebesgue integral from measure of undergraph
            - Lorentz $L^{p,q}$
            - unit mass $\mathrm{E}([0,1]) = \mathrm{E}(0,1), \mathrm{E}([-\pi,\pi]) = \mathrm{E}(S^1)$