Info

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Fixed points of functions $U \subseteq \mathbb{R} \to U$

fixed points of continuous functions $[a, b] \to [a, b]$

Example

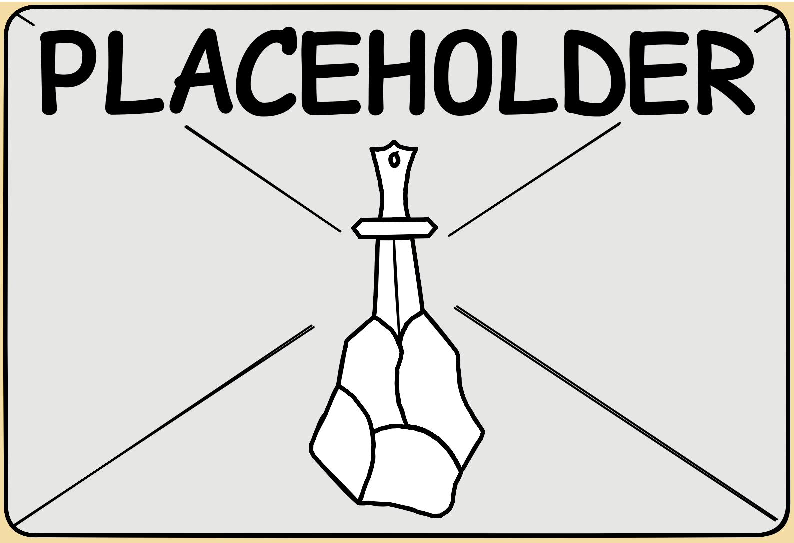
x^2

```
top=1.2
left = -0.5
right = +1.5
bottom = -0.2
---
x^2
x
```

Let

$$f : [a, b] \to [a, b]$$

be a continuous function. Then there is a point $x \in [a, b]$



such that

$$f(x) = x$$

that is, a fixed point.

- A continuous function

  $$f : [a, b] \to [a, b]$$

  means

  $$f(a), f(b) \in [a, b]$$

  Then consider the continuous function

  $$\begin{aligned} [a, b] &\to \mathbb{R} \\ x &\mapsto f(x) - x \end{aligned}$$

  - This function maps

    $$\begin{aligned} a &\mapsto f(a) - a \geq 0 \\ b &\mapsto f(b) - b \leq 0 \end{aligned}$$

  - If both values are not equal, by

    (Intermediate value theorem for continuous $[a, b] \to \mathbb{R}$) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose there are two points $\alpha < \beta$ in $[a, b]$ such that $f(\alpha) \neq f(\beta)$. Then $f(x)$ takes every value between $f(\alpha)$ and $f(\beta)$ for $x \in (\alpha, \beta)$. That is

    $$f(\alpha, \beta) \supseteq (f(\alpha), f(\beta))$$

    Let k be a number between $f(\alpha)$ and $f(\beta)$. Then apply

$f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous and suppose $f(a), f(b)$ have opposite sign, that is,

$$f(a)f(b) < 0$$

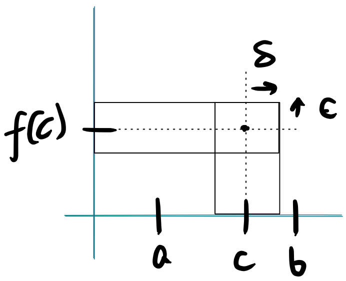
Then $\exists c \in (a,b)$ such that

$$f(c) = 0$$

- Take $f(a) > 0, f(b) < 0$.
  - Define

    $$c := \sup\{x \in [a,b] : f(x) \geq 0\}$$

    - This set has a and is bounded above by b. So c exists and $c \in (a,b)$.
  - Assume $f(c) \neq 0$
    - $\exists\delta$ such that δ–$B(c)$ has same sign as $f(c)$ from the sign property
      - If $f(c) > 0$

        then

        $$x \in (c, \delta) \implies f(x) \geq 0$$

        meaning elements of (c, δ) also belong to the above set. **Contradiction** to the definition of c.
      - Assume $f(c)[](space. R.1. f. cont.$
    - Hence, $f(c) = 0$.

for the function $g : [\alpha, \beta] \to \mathbb{R}$

$$g(x) := f(x) - k$$

Use the fact that continuous images of connected sets are connected.

- we have a point $c \in [a,b]$ where

$$f(c) - c = 0$$

- If both values are equal, then both are zero, so the function $f(x) = x$ on $[a,b]$.

Current note has 0 direct children and 0 total descendants.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - R1
            - fixed pts Fixed points of functions $U \subseteq \mathbb{R} \to U$

And it has 5 siblings.

- stamp stamp
    - Rf subobjects of and functions on $\mathbb{R}^n, T^n, S^n, \mathbb{C}^n$
        - R1
            - discont types Types of discontinuities of functions on $\mathbb{R}$
            - extremum Extremum of functions $\mathbb{R} \to \mathbb{R}$
            - fixed pts Fixed points of functions $U \subseteq \mathbb{R} \to U$
            - limit of fractions $\lim_{x \to a} \frac{f(x)}{g(x)}$ of functions $f, g : U \subseteq \mathbb{R} \to \mathbb{R}$
            - Var and sequence of functions Total variation and sequence of continuous functions